ON THE PSEUDO-RESOLVENT TO THE KERNEL OF AN INTEGRAL EQUATION*

BY

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The method developed by FREDHOLM† for the solution of a non-homogeneous linear integral equation is comparatively simple in case the corresponding homogeneous equation admits no solutions not identically zero; in terms of the determinant and the first minor for the kernel of the given equation a resolvent function is defined, by means of which the solution is readily found. If, on the other hand, the homogeneous equation admits non-zero solutions, the development of necessary and sufficient conditions for the existence of solutions of the non-homogeneous equation by FREDHOLM'S method, while similar in character to the simpler case, involves the introduction of minors of higher order, thereby necessitating somewhat tedious and complicated algebraic manipulation. Other methods of approach have been devised in this case—for example, SCHMIDT'S detour through the easier theories of equations with kernels which are symmetric, which remain small in absolute value, or which consist of the sum of a finite number of terms of the form $\varphi(x)\psi(y)$, to that of the equation with general unsymmetric kernel.‡ Such methods, although they present many points of interest in themselves, involve a considerable amount of work before arriving at the few essential facts regarding the solution of the integral equation. It is the purpose of the present paper to furnish a scheme for obtaining these essential facts rapidly and directly, without the introduction of any new concepts beyond those involved in the simpler case; by an easy artifice, the solution of a given integral equation is reduced to the solution of another, for whose kernel a resolvent function exists;

* Read before the American Mathematical Society, September 12, 1911.

† Acta Mathematica, vol. 27 (1903), p. 365.

‡ Mathematische Annalen, vol. 63 (1907), p. 459; vol. 64 (1907), p. 161. A recent paper by T. LALESCO, Bulletin de la Société Mathématique de France, vol. 39 (1911), p. 85, is also concerned with the problem here presented; the method is more complicated than that here used and the steps of the work, as well as the final result, are quite different. On the other hand, LALESCO obtains valuable information regarding the behavior of the resolvent in the neighborhood of a pole, which is not obtainable by the procedure here suggested.

a *pseudo-resolvent* is thus defined for the kernel of the given equation, and the theory is readily established. It should be noted that the facts are given completely by Theorems I, II, and III; the remainder of the paper is concerned with various subsidiary considerations.

Throughout, the known functions of one variable are supposed to be given, and the unknown to be sought, in the interval $a \leqq x \leqq b$; the kernel $K(x, y)$ is supposed given in the region $a \leqq x \leqq b$, $a \leqq y \leqq b$; these premises will not be repeated in the statement of the theorems. The treatment is restricted to the case that all functions involved are continuous.

From the elementary part of the FREDHOLM theory* we assume the following

LEMMA I. *If the kernel $K(x, y)$ is continuous, and if either of the equations*

$$\varphi(x) = \int_a^b K(x, s)\, \varphi(s)\, ds, \tag{1}$$

$$\psi(x) = \int_a^b \psi(s)\, K(s, x)\, ds, \tag{1'}$$

has no continuous solutions not identically zero, then the other has no continuous solutions not identically zero; there exists a continuous function $k(x, y)$, the resolvent to the given kernel $K(x, y)$, such that

$$k(x, y) = K(x, y) + \int_a^b K(x, s)\, k(s, y)\, ds,$$

$$k(x, y) = K(x, y) + \int_a^b k(x, s)\, K(s, y)\, ds;$$

if $f(x)$ is continuous, the equation

$$u(x) = f(x) + \int_a^b K(x, s)\, u(s)\, ds, \tag{2}$$

has one and only one continuous solution, given by the formula:

$$u(x) = f(x) + \int_a^b k(x, s) f(s)\, ds.$$

We assume also the properties of normal orthogonal sets of continuous functions,† and in particular the following:

LEMMA II. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are linearly independent continuous functions, then there exists a normal orthogonal set of continuous functions $v_1(x), v_2(x), \cdots, v_n(x)$, such that any function of either set is linearly dependent on the functions of the other set.*

We proceed to the proof of the theorems giving the desired facts.

* Cf., for instance, BÔCHER, *An Introduction to the Study of Integral Equations*, p. 29.

† BÔCHER, loc. cit., p. 52.

THEOREM I. *If the kernel $K(x, y)$ is continuous, the equations* (1), (1′) *have each a finite number of linearly independent continuous solutions, on which every other continuous solution is linearly dependent; and this number is the same for the two equations.*

The proof of the first part of the theorem by direct methods is well known,* and will not be repeated here; it is only the direct proof of the equality of the numbers of solutions which is new. Let the sets of solutions of (1), (1′) be replaced, according to Lemma II, by equivalent normal orthogonal sets $\varphi_1(x), \varphi_2(x), \cdots \varphi_n(x); \psi_1(x), \psi_2(x), \cdots, \psi_m(x)$. If their numbers are unequal, let $n < m$. Define

$$L(x, y) = K(x, y) - \sum_{i=1}^{n} \psi_i(x)\, \varphi_i(y). \tag{3}$$

It will be shown that of the two homogeneous equations

$$u(x) = \int_a^b L(x, s)\, u(s)\, ds, \tag{4}$$

$$v(x) = \int_a^b v(s)\, L(s, x)\, ds, \tag{4′}$$

one possesses a continuous solution not identically zero, while the other does not; so that Lemma I is contradicted.

On replacing $L(x, s)$ by its defining formula, we see that any continuous solution of (4) must satisfy the equation

$$u(x) = -\sum_{i=1}^{n} \psi_i(x) \int_a^b \varphi_i(s)\, u(s)\, ds + \int_a^b K(x, s)\, u(s)\, ds; \tag{5}$$

if we multiply this equation by $\psi_j(x)$ $[j = 1, 2, \cdots, n]$, and integrate with respect to x between the limits a and b, we find, on account of the normal orthogonality of the set $\psi_1(x), \psi_2(x), \cdots, \psi_n(x)$,

$$\int_a^b \varphi_j(s)\, u(s)\, ds = 0 \qquad [j = 1, 2, \cdots, n]; \tag{6}$$

hence $u(x)$ must satisfy

$$u(x) = \int_a^b K(x, s)\, u(s)\, ds, \tag{7}$$

so that

$$u(x) = \sum_{i=1}^{n} c_i\, \varphi_i(x); \tag{8}$$

but again in view of (6),

$$c_i = 0 \qquad [i = 1, 2, \cdots, n],$$

* BÔCHER, loc. cit., p. 56.

and therefore

$$u(x) = 0,$$

as was asserted.

On the other hand, at least one solution of (4′) not identically zero can be found, namely $v(x) = \psi_{n+1}(x)$; for

$$\psi_{n+1}(x) - \int_a^b \psi_{n+1}(s)\, L(s, x)\, ds = \psi_{n+1}(x) - \int_a^b \psi_{n+1}(s)\, K(s, x)\, ds + \sum_{i=1}^{n} \left[\int_a^b \psi_{n+1}(s)\, \psi_i(s)\, ds \right] \varphi_i(x) = 0.$$

Thus the assumption that $n < m$ leads to a contradiction of Lemma I, and is therefore untenable. A similar proof holds for the hypothesis $n > m$, so that the theorem is proved.

THEOREM II. *If the complete sets of normal orthogonal solutions of* (1), (1′), *are respectively* $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$; $\psi_1(x), \psi_2(x), \cdots, \psi_n(x)$, *there exists a* continuous function* $k(x, y)$, *the pseudo-resolvent to the given kernel* $K(x, y)$, *such that*

$$(9) \quad k(x, y) = K(x, y) + \int_a^b K(x, s)\, k(s, y)\, ds - \sum_{i=1}^{n} \psi_i(x)\, \psi_i(y),$$

$$(9') \quad k(x, y) = K(x, y) + \int_a^b k(x, s)\, K(s, y)\, ds - \sum_{i=1}^{n} \varphi_i(x)\, \varphi_i(y).$$

To prove this, we carry over from the proof of Theorem I the definition (3) of $L(x, y)$ and the property that the equation (4) has no continuous solution not identically zero. Let us construct the *resolvent* $k(x, y)$ to the kernel $L(x, y)$; its existence is assured by Lemma I, which also furnishes the identities:

$$(10) \quad k(x, y) = L(x, y) + \int_a^b L(x, s)\, k(s, y)\, ds,$$

$$(10') \quad k(x, y) = L(x, y) + \int_a^b k(x, s)\, L(s, y)\, ds.$$

The first of these identities may be rewritten in the form:

$$(11) \quad k(x, y) = K(x, y) - \sum_{i=1}^{n} \psi_i(x)\, \varphi_i(y) + \int_a^b K(x, s)\, k(s, y)\, ds - \sum_{i=1}^{n} \psi_i(x) \int_a^b \varphi_i(s)\, k(s, y)\, ds;$$

* In fact, an infinite number of such functions; their relations to one another are mentioned later.

on multiplying by $\psi_j(x)$ $[j = 1, 2, \cdots, n]$, and integrating with respect to x from a to b, we obtain, after easy simplifications,

$$\int_a^b \varphi_j(s)\, k(s, y)\, ds = \psi_j(y) - \varphi_j(y) \quad [j = 1, 2, \cdots, n];$$

and the substitution of this result in (11) yields at once (9), the first of the formulæ to be proved. The second formula may be verified in like manner.

THEOREM III. *A necessary and sufficient condition that the equation*

$$u(x) = f(x) + \int_a^b K(x, s)\, u(s)\, ds \tag{2}$$

(where $f(x)$ is continuous) have a continuous solution, is that

$$\int_a^b \psi_i(x) f(x)\, dx = 0 \qquad [i = 1, 2, \cdots, n]. \tag{12}$$

If this condition is fulfilled, any continuous solution may be written in the form

$$u(x) = f(x) + \int_a^b k(x, s) f(s)\, ds + \sum_{i=1}^{n} c_i \varphi_i(x), \tag{13}$$

where $c_1, c_2, \cdots, c_n$ are constants.

The theorem may be paraphrased as follows: if (2) has a continuous solution, it must be given by (13); the continuous function defined by (13) is a solution of (2) when and only when conditions (12) are fulfilled. The proof of these two assertions is conducted by the use of the pseudo-resolvent and the identities (9), (9′), in exactly the same fashion as the proof of the corresponding part of Lemma I is ordinarily given by use of the resolvent and its characteristic properties. If we multiply (2) by $k(z, x)$ and integrate with respect to x, we infer, by use of (9′), that

$$\int_a^b K(z, s)\, u(s)\, ds = \int_a^b k(z, x) f(x)\, dx + \sum_{i=1}^{n} \varphi_i(z) \int_a^b \varphi_i(s)\, u(s)\, ds;$$

the direct substitution of this result in (2) leads to:

$$u(x) = f(x) + \int_a^b k(x, s) f(s)\, ds + \sum_{i=1}^{n} \varphi_i(x) \int_a^b \varphi_i(s)\, u(s)\, ds,$$

which is of the form (13), with

$$c_i = \int_a^b \varphi_i(s)\, u(s)\, ds \qquad [i = 1, 2, \cdots, n].$$

On the other hand, suppose that (13) is satisfied; a similar process of multiplication by $K(z, x)$, integration with respect to x, use of (9), and substitution in (13), leads to the result:

$$u(x) = f(x) + \int_a^b K(x, s)\, u(s)\, ds - \sum_{i=1}^{n} \psi_i(x) \int_a^b \psi_i(s) f(s)\, ds.$$

Thus the function $u(x)$ defined by (13) will actually be a solution of (2) when and, on account of the linear independence of $\psi_1(x), \psi_2(x), \cdots, \psi_n(x)$, only when conditions (12) are satisfied. Our theorem is thus proved.

It is evident that the function $k(x, y)$ defined in the course of the proof of Theorem II is not uniquely defined by the relations (9), (9′); these are satisfied by every function of the form

$$k(x, y) + \sum_{i=1, j=1}^{n, n} c_{i,j}\, \varphi_i(x)\, \psi_j(y),$$

(where $c_{i,j}$ are constants) and by no other continuous functions. The particular function which arose in the proof may be completely specified, if this is desired, by demanding, in addition to (9) and (9′) the validity of one of the relations

$$\int_a^b \varphi_i(s)\, k(s, x)\, ds = \psi_i(x) - \varphi_i(x) \quad [i = 1, 2, \cdots, n].$$

$$\int_a^b k(x, s)\, \psi_i(s)\, ds = \varphi_i(x) - \psi_i(x) \quad [i = 1, 2, \cdots, n].$$

It is natural to ask whether functions of still more general character could serve the same purpose—functions which do not necessarily satisfy (9), (9′), but which nevertheless suffice to give the solutions of (2) in the form (13) under conditions (12). *Any continuous function $l(x, y)$ will be called a pseudo-resolvent to the kernel $K(x, y)$ if for every continuous function $f(x)$ satisfying conditions* (12), *every continuous solution of* (2) *is expressible in the form*:

$$u(x) = f(x) + \int_a^b l(x, s) f(s)\, ds + \sum_{i=1}^{n} c_i\, \varphi_i(x) \tag{14}$$

and every function of the form (14) *is a solution of* (2). We then have

THEOREM IV. *In order that the continuous function $l(x, y)$ be a pseudo-resolvent, it is necessary and sufficient that*

$$(15)\quad l(x, y) = K(x, y) + \int_a^b K(x, s)\, l(s, y)\, ds - \sum_{i=1}^{n} \Psi_i(x)\, \psi_i(y);$$

$$(15')\quad l(x, y) = K(x, y) + \int_a^b l(x, s)\, K(s, y)\, ds - \sum_{i=1}^{n} \varphi_i(x)\, \Phi_i(y);$$

where $\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)$; $\Psi_1(x), \Psi_2(x), \cdots, \Psi_n(x)$ are continuous functions such that

$$(16)\quad \int_a^b \varphi_i(x)\, \Phi_j(x)\, dx = \left\{ \begin{matrix} 0, & i \neq j \\ 1, & i = j \end{matrix} \right\};$$

$$(16')\quad \int_a^b \Psi_i(x)\, \psi_j(x)\, dx = \left\{ \begin{matrix} 0, & i \neq j \\ 1, & i = j \end{matrix} \right\} \quad [i, j = 1, 2, \cdots, n].$$

By way of proof, we assume that $l(x, y)$ is a pseudo-resolvent and write

$$Q(x, y) = l(x, y) - K(x, y) - \int_a^b K(x, s)\, l(s, y)\, ds.$$

It is clear that

$$\int_a^b \psi_i(s)\, Q(s, x)\, ds = -\psi_{i_s}(x) \qquad [i = 1, 2, \cdots, n],$$

and that if we define

$$\Psi_i(x) = -\int_a^b Q(x, s)\, \psi_i(s)\, ds \qquad [i = 1, 2, \cdots, n],$$

the relations (16′) are satisfied. Now by our definition of a pseudo-resolvent the function

$$u(x) = f(x) + \int_a^b l(x, s) f(s)\, ds$$

must verify the equation

$$u(x) = f(x) + \int_a^b K(x, s)\, u(s)\, ds$$

for every $f(x)$ satisfying (12). Carrying out the process of substitution in the equation, we obtain, after trifling reductions, the condition

$$\int_a^b Q(x, s) f(s)\, ds = 0,$$

valid for every such function $f(x)$. But such a function is given by the formula

$$f(s) = Q(x, s) + \sum_{i=1}^{n} \Psi_i(x)\, \psi_i(s)$$

for any value of x; hence we have the identity

$$\int_a^b Q(x, s)\,[Q(x, s) + \sum_{i=1}^{n} \Psi_i(x)\, \psi_i(s)]\, ds = 0.$$

As however evidently

$$\int_a^b \psi_j(s)\,[Q(x, s) + \sum_{i=1}^{n} \Psi_i(x)\, \psi_i(s)]\, ds = 0 \quad [j = 1, 2, \cdots, n],$$

we have on multiplying the latter equations by $\Psi_j(x)$ and adding to the preceding

$$\int_a^b [Q(x, s) + \sum_{i=1}^{n} \Psi_i(x)\, \psi_i(s)]^2\, ds = 0,$$

so that

$$Q(x, s) = -\sum_{i=1}^{n} \Psi_i(x)\, \psi_i(s),$$

and (15) is demonstrated.

The same plan cannot be used in the proof of (15′);* we shall deduce this formula from (15), making use of the properties of the simpler pseudo-resolvent $k(x, y)$. Operating with $k(x, y)$ in the usual manner on (15) as if it were an equation to be solved for an unknown function $l(x, y)$, we deduce that

$$l(x, y) = K(x, y) - \sum_{i=1}^{n} \Psi_i(x)\, \psi_i(y) + \int_a^b k(x, s)\, K(s, y)\, ds$$

$$- \sum_{i=1}^{n} \left[\int_a^b k(x, s)\, \Psi_i(s)\, ds \right] \psi_i(y) + \sum_{i=1}^{n} \varphi_i(x) \int_a^b \varphi_i(s) l(s, y) ds,$$

or, by (9′),

$$l(x, y) = k(x, y) + \sum_{i=1}^{n} \varphi_i(x) \left[\varphi_i(y) + \int_a^b \varphi_i(s)\, l(s, y)\, ds \right]$$

$$- \sum_{i=1}^{n} \left[\Psi_i(x) + \int_a^b k(x, s)\, \Psi_i(s)\, ds \right] \psi_i(y).$$

* The same proof *would* of course apply if we assumed as part of the definition of a pseudo-resolvent that its connection with the kernel remains unaltered under interchange of the two variables. From the standpoint of the logic of mathematics some interest attaches to the fact that it is unnecessary to make this assumption.

If we multiply this equation by $K(y, z)$ and integrate with respect to y between the limits a and b, making use again of (9′), we find that

$$\int_a^b l(x, y) K(y, z)\, dy = k(x, z) - K(x, z) + \sum_{i=1}^{n} \varphi_i(x) \Big[\varphi_i(z) + \int_a^b \varphi_i(y) K(y, z)\, dy + \int_a^b \int_a^b \varphi_i(s) l(s, y) K(y, z)\, dy\, ds \Big] - \sum_{i=1}^{n} \Big[\Psi_i(y) + \int_a^b k(y, s) \Psi_i(s)\, ds \Big] \psi_i(z),$$

so that

$$l(x, z) - K(x, z) - \int_a^b l(x, y) K(y, z)\, dy = \sum_{i=1}^{n} \varphi_i(x) \Big[\int_a^b \varphi_i(s) l(s, z)\, ds - \int_a^b \varphi_i(s) K(s, z)\, ds - \int_a^b \int_a^b \varphi_i(s) l(s, y) K(y, z)\, dy\, ds \Big].$$

This is of the form (15′), provided we set

$$-\Phi_i(z) = \int_a^b \varphi_i(s) l(s, z)\, ds - \int_a^b \varphi_i(s) K(s, z)\, ds - \int_a^b \int_a^b \varphi_i(s) l(s, y) K(y, z)\, dy\, ds \qquad [i = 1, 2, \cdots, n]$$

and it is evident that with this choice of $\Phi_i(z)$, the restrictions (16′) are also satisfied.

We have thus shown the necessity of the conditions stated for a pseudo-resolvent; that they are sufficient may be verified at once by following out exactly the lines of the proof of Theorem III.

To complete the theory of these more general pseudo-resolvents, we add a sort of partial converse of Theorem IV:

THEOREM V. *For any set of continuous functions* $\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x); \Psi_1(x), \Psi_2(x), \cdots, \Psi_n(x)$, *satisfying* (16), (16′), *there exist pseudo-resolvents* $l(x, y)$ *satisfying* (15), (15′).

To construct them we have only to solve (15), (15′) for $l(x, y)$ by the use of any special pseudo-resolvent (such as $k(x, y)$ of Theorems II and III),

and to pick out a solution common to the two. Such a solution is

$$l(x, y) = k(x, y) - \sum_{i=1}^{n} \varphi_i(x)\left[\Phi_i(y) + \int_a^b \Phi_i(s)\, k(s, y)\, ds\right] - \sum_{i=1}^{n}\left[\Psi_i(x) + \int_a^b k(x, s)\, \Psi_i(s)\, ds\right]\psi_i(y);$$

the general solution is obtained by adding to this special one:

$$\sum_{i=1, j=1}^{n, n} c_{i, j}\, \varphi_i(x)\, \psi_j(y),$$

where $c_{i, j}$ are constants.

The question as to the most general possible pseudo-resolvent is thus completely answered by Theorems IV and V. The pseudo-resolvent $l(x, y)$ reduces to the simpler one $k(x, y)$ in case we choose

$$\Phi_i(x) = \varphi_i(x), \tag{17}$$

$$\Psi_i(x) = \psi_i(x) \qquad [i = 1, 2, \cdots, n]. \tag{17'}$$

A case in which the second kind of pseudo-resolvent differs from the first is obtained on replacing (17) by

$$\Phi_i(x) = \int_a^b \varphi_i(s)\, K(s, x)\, ds \qquad [i = 1, 2, \cdots, n]$$

or (17′) by

$$\Psi_i(x) = \int_a^b K(x, s)\, \psi_i(s)\, ds \qquad [i = 1, 2, \cdots, n]$$

or both. Out of the large number of possible pseudo-resolvents, that of theorem II especially recommends itself both by the symmetry of its characteristic formulæ (9), (9′), and by the ease with which its existence can be proved directly.

It is of interest to investigate from the present standpoint the pseudo-resolvent which presents itself in Fredholm's treatment;* it must of course fall under the case of Theorem IV for some choice of $\Phi_1, \Phi_2, \cdots, \Phi_n; \Psi_1, \Psi_2, \cdots, \Psi_n$. If we remove from a statement of the properties of the Fredholm pseudo-resolvent all reference to the successive minors, in terms of which it is derived, we obtain the following essential facts:† *There exist n points*

* FREDHOLM, loc. cit., p. 374. See also the very clear and succinct presentation by HORN, Einführung in die Theorie der partiellen Differentialgleichungen p. 199.

† HORN, loc. cit., p. 209.

$x_1, x_2, \cdots, x_n$, and n continuous solutions $\bar{\varphi}_1(x), \bar{\varphi}_2(x), \cdots, \bar{\varphi}_n(x)$ of* (1), *such that*

$$\bar{\varphi}_i(x_j) = \left\{ \begin{matrix} 0, & i \neq j \\ 1, & i = j \end{matrix} \right\} \qquad [i, j = 1, 2, \cdots, n]; \tag{18}$$

and there exist n points $y_1, y_2, \cdots, y_n$, and n continuous solutions $\bar{\psi}_1(x)$, $\bar{\psi}_2(x), \cdots, \bar{\psi}_n(x)$ of* (1'), *such that*

$$\psi_i(y_j) = \left\{ \begin{matrix} 0, & i \neq j \\ 1, & i = j \end{matrix} \right\} \qquad [i, j = 1, 2, \cdots, n]. \tag{18'}$$

The Fredholm pseudo-resolvent $l(x, y)$ satisfies the relations:

$$l(x, y) = K(x, y) + \int_a^b K(x, s) l(s, y)\, ds - \sum_{i=1}^{n} K(x, y_i) \bar{\psi}_i(y), \tag{19}$$

$$l(x, y) = K(x, y) + \int_a^b l(x, s) K(s, y)\, ds - \sum_{i=1}^{n} \bar{\varphi}_i(x) K(x_i, y). \tag{19'}$$

In order to bring this into connection with our former work, we make first the following observation: the necessary and sufficient condition on the points $x_1, x_2, \cdots, x_n$ for the existence of the solutions $\bar{\varphi}_1(x), \bar{\varphi}_2(x), \cdots, \bar{\varphi}_n(x)$ is that

$$\Delta = \begin{vmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \cdot & \cdot & \cdots & \cdot \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{vmatrix} \neq 0. \tag{20}$$

That this condition is necessary appears from the fact that if $\bar{\varphi}_1(x), \bar{\varphi}_2(x), \cdots, \bar{\varphi}_n(x)$ have the properties (18), they must then be connected with $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$ by the relations

$$\begin{aligned} \varphi_1(x) &= \varphi_1(x_1)\, \bar{\varphi}_1(x) + \varphi_1(x_2)\, \varphi_2(x) + \cdots + \varphi_1(x_n)\, \bar{\varphi}_n(x), \\ \varphi_2(x) &= \varphi_2(x_1)\, \bar{\varphi}_1(x) + \varphi_2(x_2)\, \bar{\varphi}_2(x) + \cdots + \varphi_2(x_n)\, \bar{\varphi}_n(x), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_n(x) &= \varphi_n(x_1)\, \bar{\varphi}_1(x) + \varphi_n(x_2)\, \bar{\varphi}_2(x) + \cdots + \varphi_n(x_n)\, \bar{\varphi}_n(x), \end{aligned} \tag{21}$$

* The sets of points $x_1, x_2, \cdots, x_n$; $y_1, y_2, \cdots, y_n$ may be chosen in a great variety of ways; indeed they are subject only to the condition of yielding a value different from zero on being substituted for the $2n$ arguments of the earliest minor which does not vanish identically; there is thus a correspondingly great freedom of choice in the selection of the FREDHOLM pseudo-resolvent.

and the vanishing of Δ would contradict the linear independence of $\varphi_1(x)$, $\varphi_2(x), \cdots, \varphi_n(x)$. On the other hand, the condition is sufficient, the values of $\bar{\varphi}_1(x), \bar{\varphi}_2(x), \cdots, \bar{\varphi}_n(x)$ being given by the solution of the equations (21).

In the next place, there will certainly exist some set of points $x_1, x_2, \cdots, x_n$ rendering $\Delta \neq 0$; for the identical vanishing of Δ for all choices of $x_1, x_2, \cdots, x_n$ would involve the linear dependence of $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$.*

If now (19′) and (15′) are to be in agreement, we must have

$$\sum_{i=1}^{n} \varphi_i(x)\,\Phi_i(y) = \sum_{i=1}^{n} \bar{\varphi}_i(x)\,K(x_i, y); \tag{22}$$

substituting here for $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$ the values given by (21), we may write this in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \varphi_i(x_j)\,\bar{\varphi}_j(x)\,\Phi_i(y) = \sum_{i=1}^{n} \bar{\varphi}_i(x)\,K(x_i, y),$$

or, giving to x the special value x_p $[p = 1, 2, \cdots, n]$,

$$\sum_{i=1}^{n} \varphi_i(x_p)\,\Phi_i(y) = K(x_p, y) \qquad [p = 1, 2, \cdots, n].$$

These equations suffice to determine $\Phi_i(y)$ as being equal to $1/\Delta$ multiplied into the determinant obtained on replacing the ith column of Δ by $K(x_1, y)$, $K(x_2, y), \cdots, K(x_n, y)$. The functions $\Phi_i(y)$ thus obtained do actually satisfy (22), thus identifying (19′) with (15′); they give these relations the form:†

$$l(x, y) = K(x, y) + \int_a^b l(x, s)\,K(s, y)\,ds$$

$$+\frac{1}{\Delta}\begin{vmatrix} 0 & \varphi_1(x) & \varphi_2(x) & \cdots & \varphi_n(x) \\ K(x_1, y) & \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ K(x_2, y) & \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K(x_n, y) & \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{vmatrix};$$

finally, without any further restrictions, they satisfy (16). To exemplify

* This theorem is stated incidentally in a note by RICHARDSON and HURWITZ, Bulletin of the American Mathematical Society, vol. 16 (1909–10), p. 18. It is stated explicitly and proved by CURTISS, Bulletin of the American Mathematical Society, vol. 17 (1910–11), p. 463.

† In the final statement of the theorem, this form is further condensed by the absorption of the term $K(x, y)$ into the determinant.

the latter statement, let us write out the value, say of $\Phi_1(y)$:

$$\Phi_1(y) = \frac{1}{\Delta}\begin{vmatrix} K(x_1, y) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ K(x_2, y) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ K(x_n, y) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{vmatrix},$$

multiply by $\varphi_i(y)$ $[i = 1, 2, \cdots, n]$, and integrate. Since

$$\int_a^b K(x_j, y)\,\varphi_i(y)\,dy = \varphi_i(x_j) \quad [i, j = 1, 2, \cdots, n],$$

this process shows at once that

$$\int_a^b \varphi_i(y)\,\Phi_1(y)\,dy = 0 \qquad [i = 2, 3, \cdots, n],$$

$$\int_a^b \varphi_1(y)\,\Phi_1(y)\,dy = 1;$$

a similar proof applies to $\Phi_2(y), \cdots, \Phi_n(y)$.

Of course, the same reasoning could be used for the identification of the formulæ (16), (19). We state the result as follows:

THEOREM VI. *There exist sets of points* $x_1, x_2, \cdots, x_n$; $y_1, y_2, \cdots, y_n$, *such that*

$$\Delta = \begin{vmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{vmatrix} \neq 0,$$

$$\Delta_1 = \begin{vmatrix} \psi_1(y_1) & \psi_2(y_1) & \cdots & \psi_n(y_1) \\ \psi_1(y_2) & \psi_2(y_2) & \cdots & \psi_n(y_2) \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ \psi_1(y_n) & \psi_2(y_n) & \cdots & \psi_n(y_n) \end{vmatrix} \neq 0.$$

A Fredholm pseudo-resolvent is obtained by selecting any such set, and choosing for $\Phi_i(y)$ *the constant* $1/\Delta$ *multiplied into the determinant obtained on replacing the* i*th column of* Δ *by* $K(x_1, y), K(x_2, y), \cdots, K(x_n, y)$; *and for* $\Psi_i(x)$ *the constant* $1/\Delta_1$ *multiplied into the determinant obtained on replacing the* i*th*

column of Δ_1 *by* $K(x, y_1), K(x, y_2), \cdots, K(x, y_n)$. *Such a pseudo-resolvent satisfies the relations*:

$$l(x, y) = \int_a^b K(x, s)\, l(s, y)\, ds + \frac{1}{\Delta_1} \begin{vmatrix} K(x, y) & \psi_1(y) & \psi_2(y) & \cdots & \psi_n(y) \\ K(x, y_1) & \psi_1(y_1) & \psi_2(y_1) & \cdots & \psi_n(y_1) \\ K(x, y_2) & \psi_1(y_2) & \psi_2(y_2) & \cdots & \psi_n(y_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K(x, y_n) & \psi_1(y_n) & \psi_2(y_n) & \cdots & \psi_n(y_n) \end{vmatrix},$$

$$l(x, y) = \int_a^b l(x, s)\, K(s, y)\, ds + \frac{1}{\Delta} \begin{vmatrix} K(x, y) & \varphi_1(x) & \varphi_2(x) & \cdots & \varphi_n(x) \\ K(x_1, y) & \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ K(x_2, y) & \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K(x_n, y) & \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{vmatrix}.$$

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INFINITE SYSTEMS OF INDIVISIBLE GROUPS*

BY

G. A. MILLER

§ 1. *Introduction.*

A group is said to be *divisible* (zerfallend) if it is the direct product of two groups, neither of which is the identity. If it is not such a direct product it is said to be *indivisible* or prime.† Among the indivisible groups the simple groups occupy the most important place. Divisible groups are necessarily composite but there are also many types of composite groups which are not divisible. Each indivisible group may be used to construct an infinite system of distinct divisible groups by forming successive direct products with this group as a factor. That is, by forming the direct product of two groups which are simply isomorphic with the given group H, and then forming the direct product of a group which is simply isomorphic with H and the direct product just found, etc. Such successive direct products we shall call successive *powers of the given group.* The two infinite systems formed by the successive powers of two distinct indivisible groups have no group in common.‡

FROBENIUS and STICKELBERGER proved that the cyclic *primary* groups, that is, the cyclic groups whose orders are powers of a single prime number, are the only abelian groups which are indivisible.§ Hence every divisible abelian group is a direct product of cyclic primary groups and every such direct product is an abelian group. Moreover, every abelian group is completely determined by its cyclic primary factor groups. In the articles cited, MACLAGAN-WEDDERBURN and REMAK extended this theorem by proving that every divisible group is completely determined by the factor groups when ever it is represented as a direct product of indivisible groups.

As there is only one indivisible abelian group of a given order an abelian indivisible group can be completely defined by giving its order. The infinite system of indivisible abelian groups will be represented by the symbol S_1. The main object of the present paper is to study infinite systems of indivisible groups which are such that no two distinct groups of the same system have a

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† MACLAGAN-WEDDERBURN, Annals of Mathematics, series 2, vol. 10 (1909), p. 173.

‡ REMAK, Crelle's Journal, vol. 139 (1911), p. 293.

§ FROBENIUS and STICKELBERGER, Crelle's Journal, vol. 86 (1879), p. 217.

common order. It is hoped that this classification will serve to exhibit more clearly some fundamental common properties of known categories of groups and also to extend our knowledge as regards the groups which have such common properties. The rapid increase in the number of known groups calls for improvements of the methods of classification and for a determination of common properties which may serve as a suitable basis of classification. As each indivisible group may be used to construct an infinite system of divisible groups by forming its successive powers, and as the two infinite systems of such groups which may be derived from two distinct indivisible groups have no group in common, it is clear that infinite systems of indivisible groups afford large possibilities for group construction.

In what follows we shall direct attention to a still larger source for group construction by showing how we may construct an infinite system of distinct *indivisible* groups by starting with any given non-abelian indivisible group, and by proving that two of these systems cannot have any group in common unless one of the two systems is contained entirely in the other system. We shall also obtain a formula by means of which the number of operators of highest order in any one of such an infinite system of indivisible groups can be readily determined. These developments call for a determination of adtional general theorems relating to divisible groups. Several such theorems are given in the following section and these constitute perhaps the most important part of the present paper.

Some of the best known infinite categories of non-abelian groups are so elementary as to offer little interest from the standpoint of group construction. For instance, the Hamiltonian groups are merely direct products of the quaternion group and powers of groups of S_1, and every such direct product is Hamiltonian provided either all the factors selected from S_1 are of odd order or the group of order 2 constitutes the only factor of even order selected from S_1.* Hence the quaternion group is the only indivisible group among the totality of Hamiltonian groups. A closely related infinite category of groups is the one formed by the direct products of the octic group and powers of the group of order 2. There is one and only one such group of order 2^α, $\alpha > 2$, and it has been proved that this infinite system of groups is composed of all the non-abelian groups in which no more than one fourth of the operators have orders larger than 2.†

Two important infinite systems of indivisible groups received attention very early as a result of their prominent position in the theory of substitution groups; viz., the symmetric and the alternating groups. The systems of

* MILLER, Bulletin of the American Mathematical Society, vol. 4 (1898), p. 514.

† MILLER, Annals of Mathematics, ser. 2, vol. 7 (1906), p. 55.

abstract groups which are separately simply isomorphic with these substitution groups will be denoted by S_2 and S_3 respectively. Each of these two systems has its first group in common with S_1, but otherwise no two of the three infinite systems S_1, S_2, S_2 have a common group. System S_3 is especially important, since all of its groups, with the exception of the one of order 12, are simple. General abstract definitions of the groups of these two infinite systems were first given by MOORE,* and their Sylow subgroups have been studied by RADZIG † and by FINDLAY.‡

As none of the groups of systems S_2 and S_3 contains a subgroup of index 2 which is a direct product of two simply isomorphic groups, it results from the theorem to which we referred above that each of the groups in these two systems, except the first group, can be used to construct an infinite system of indivisible groups, and that no two of these systems have a common group. In fact, it will appear that the first group of S_2 may also be used in this way but that the first one of S_3 does not have this property.

§ 2. *Some properties of divisible groups and of their subgroups.*

Suppose that the non-abelian group G is the direct product of the λ subgroups $H_1, H_2, \cdots, H_\lambda$. It is evident that the group of inner isomorphisms of G is the direct product of the groups of inner isomorphisms of $H_1, H_2, \cdots, H_\lambda$; that the commutator subgroup of G is the direct product of the commutator subgroups of $H_1, H_2, \cdots, H_\lambda$; that the central of G is the direct product of the centrals of $H_1, H_2, \cdots, H_\lambda$; and that each of the largest abelian subgroups of G is a direct product of largest abelian subgroups from each of the factor groups $H_1, H_2, \cdots, H_\lambda$. If $S^{(\alpha)}$ represents the various operators of H_α, $1 \leqq \alpha \leqq \lambda$, every operator of G is of the form

$$S^{(1)} S^{(2)} \cdots S^{(\lambda)}.$$

In any subgroup K of G the operators $S^{(\alpha)}$, where α has any one of the values $1, 2, \cdots, \lambda$, constitute a subgroup§ of H_α, and K is formed by certain isomorphisms between these constituent subgroups, if we include under the term isomorphism direct products even when one or more of the factors become the identity. A necessary and sufficient condition that K has only the identity in common with each of the factor groups $H_1, H_2, \cdots, H_\lambda$ is that it is simply isomorphic with at least two of these constituent subgroups. Hence we have the following theorem: *If K is an invariant subgroup under G and has*

* Proceedings of the London Mathematical Society, vol. 28 (1896), p. 363; cf. DICKSON, *Linear Groups*, 1901, p. 287.

† *Dissertation*, Berlin, 1895.

‡ These Transactions, vol. 5 (1904), p. 263.

§ The term subgroup is used here, and in what follows, in its most general sense, including both the identity, and the entire group.

only the identity in common with each of the factor subgroups $H_1, H_2, \cdots, H_\lambda$, *then each operator of* K *is invariant under* G. That is, if an invariant subgroup of G involves no operator besides the identity from any one of the factor subgroups $H_1, H_2, \cdots, H_\lambda$, it must be contained in the central of G, and it must be simply isomorphic with subgroups in the centrals of at least two of the factor subgroups $H_1, H_2, \cdots, H_\lambda$. As a special case of this theorem we have the known theorem that every invariant subgroup of G which has only the identity in common with each of the factor subgroups $H_1, H_2, \cdots, H_\lambda$ is abelian.*

The italicized theorem of the preceding paragraph may readily be deduced from the fact that the cross-cut† of any invariant subgroup K of G and of a factor subgroup H_α is an invariant subgroup of H_α, and the quotient group of the constituent of K contained in H_α, which corresponds to this cross-cut, is composed of invariant operators under G. This fact results directly from the theorem that the cross-cut of two invariant subgroups of any group includes all the commutators which can be formed when the two elements of each one of these commutators are selected from both of these two invariant subgroups. The italicized theorem of the preceding paragraph may also be regarded as a special case of the following theorem: *The cross-cut of the commutator subgroup of* G *and any other invariant subgroup of* G *is the direct product of subgroups of the commutator subgroups of the factor groups* $H_1, H_2, \cdots, H_\lambda$.

Since the group of inner isomorphisms of a group is necessarily non-cyclic it results that only one of the factor groups $H_1, H_2, \cdots, H_\lambda$ is non-abelian whenever the group of inner isomorphisms of G involves a cyclic subgroup of prime index. This must also be the case whenever the order of the group of inner isomorphisms of G is the product of less than four prime factors. If the order of any non-abelian solvable group G' is divisible by a prime p which does not divide the order of its group of inner isomorphisms then G' is the direct product of its Sylow subgroup whose order is a power of p and a subgroup whose order is prime to p. The truth of this theorem becomes evident if we recall that to the identity in the group of inner isomorphisms there corresponds the central of G', and hence this central is a direct product having the given Sylow subgroup as a factor. As the group of inner isomorphisms is solvable and its order is prime to p the theorem is established.‡

It was observed above that if a group is the direct product of two or more non-abelian groups its group of inner isomorphisms is also a direct product, but it is evidently not true that every group whose group of inner isomorphisms is a direct product is itself such a product. It follows from the preceding paragraph, however, that a necessary and sufficient condition that any group

* Cf. MACLAGAN-WEDDERBURN, Annals of Mathematics, series 2, vol. 10 (1909), p. 174.

† The totality of operators common to two groups is called their cross-cut.

‡ These Transactions, vol. 1 (1900), p. 66.

is the direct product of its Sylow subgroups is that its group of inner isomorphisms is the direct product of its Sylow subgroups whenever it involves more than one such subgroup. This theorem follows also directly from the fact that a necessary and sufficient condition that a group is the direct product of its Sylow subgroups is that we arrive at the identity by forming the successive groups of inner isomorphisms.* This theorem may be regarded as a special case of the following theorem. *A necessary and sufficient condition that a solvable group be a direct product of a Sylow subgroup and another subgroup is that its group of inner isomorphisms involves the corresponding Sylow subgroup as a factor of a direct product whenever it involves such a Sylow subgroup.* This theorem includes the theorem of the preceding paragraph and may be proved in an exactly similar manner.

If the order of G is a power of a single prime number the orders of each of its factor subgroups $H_1, H_2, \cdots, H_\lambda$ must be powers of the same prime. As the central of each of these factor groups exceeds unity it follows that the central of G has at least λ invariants. Hence it results that *a group of order p^m, p being any prime number, whose central is cyclic must be indivisible.* A Sylow subgroup of order p^m in the holomorph of any abelian group of order p^a has exactly p invariant operators and hence such a subgroup is always indivisible. If the holomorph of an abelian group of order p^a were divisible it would therefore be a direct product of a group whose order is prime to p and of a group involving a Sylow subgroup of order p^m. This is clearly impossible, since the operators of the former could not be commutative with each operator of the latter. As the holomorph of any abelian group is the direct product of the holomorphs of its Sylow subgroups † we have the theorem: *A necessary and sufficient condition that the holomorph of an abelian group be divisible is that the order of this abelian group is divisible by at least two different prime factors.*

If a divisible group involves an indivisible Sylow subgroup of order p^m then there is one and only one factor subgroup whose order is divisible by p. Hence it results that a divisible group cannot involve an indivisible Sylow subgroup which involves operators which are non-commutative with all the operators of the group whose orders are prime to the order of this Sylow subgroup. For instance, a dihedral group whose order is the double of an odd number is indivisible since it involves operators of order 2 which are non-commutative with all of its operators of odd order and the Sylow subgroups of order 2 are indivisible.

If a primitive substitution group is a direct product it must be the direct product of two simply isomorphic simple groups of composite order.‡ Hence

* Loewy, Mathematische Annalen, vol. 55 (1902), p. 69.

† These Transactions, vol. 4 (1903), p. 153.

‡ Maillet, *Thèse*, Paris, 1892, p. 31.

it results that when $\lambda > 2$ every maximal subgroup of G either is invariant or contains an invariant subgroup of G, besides the identity. This theorem remains true when $\lambda = 2$ provided G is not the direct product of two simply isomorphic simple groups of composite order.

§ 3. *Infinite system of indivisible groups obtained by extending given groups.*

Suppose that G is the direct product of two simply isomorphic indivisible non-abelian groups H_1, H_2. We may extend G, so as to obtain a group G' whose order is twice the order of G, by means of an operator of order 2 which transforms the corresponding operators of H_1 and H_2 into each other. We shall prove that this extended group is also non-abelian and indivisible. If G' were divisible it would be the direct product of two groups H_1', H_2'. The subgroup G of index 2 under G' would then be either the direct product of one of these factor groups of G and of a subgroup of index 2 contained in the other or it would be formed by an isomorphism between H_1' and H_2' with respect to quotient groups of order 2. In the former case G would be the direct product of two subgroups H_1'', H_2'' which are simply isomorphic with H_1 and H_2 respectively since these factor groups are indivisible.

As all the operators of one of the subgroups H_1'', H_2'' are commutative with some of the operators of G' which are not in G, it results that this subgroup, say H_1'', must be formed by establishing a simple isomorphism between H_1 and H_2. As only invariant operators of G are commutative with every operator of H_1'', it follows that H_2'' is abelian. As this is contrary to the hypothesis it results that G must be formed by establishing an isomorphism, with respect to a quotient group of order 2, between H_1' and H_2', if G' is divisible. This hypothesis implies that G contains a subgroup of index 2 which is a direct product of two groups H_1''', H_2'''. All the operators of each of these groups, taken separately, are commutative with some operators of G' which are not contained in G. Hence H_1''' and H_2''' are formed by establishing a simple isomorphism either between H_1 and H_2 or between subgroups of these groups. This must be a simple isomorphism between H_1 and H_2 in one case, as the direct product of H_1''' and H_2''' could otherwise not be of index 2 under G. If H_1''' were a simple isomorphism between H_1 and H_2 and H_2''' were a simple isomorphism between subgroups of index 2 contained in H_1 and H_2 respectively, it would clearly not be possible for all the operators of H_1''' to be commutative with all those of H_2'''. Hence we have arrived at a contradiction by assuming that G' is divisible, and it must therefore always be indivisible.

If we suppose that the common order of H_1 and H_2 is h_1 it results that the order of G' is $2h_1^2$. As we can extend the square of G' in a similar way, since G' satisfies the conditions imposed on H_1, it results that we obtain by this

process an infinite system of indivisible groups of order $2^{2n-1} h_1^{2n}$, n being an arbitrary integer $\gtreqless 0$ and h_1 being the order of H_1. The distinct factors of composition of each of these groups are the same as those of H_1 whenever one of the factors of composition of H_1 is 2. If this condition is not satisfied, 2 must be added to the factors of composition of H_1 to obtain the distinct factors of composition of this infinitive system of indivisible groups. Some of these results may be expressed in the form of a theorem as follows: *If we extend the square of any non-abelian indivisible group by means of an operator of order 2 which transforms the corresponding operators of the factors of this square into each other, we obtain another non-abelian indivisible group.* By repeating this operation we may therefore construct, by starting with any non-abelian indivisible group, an infinite system of such groups, which is such that no two of the groups of the system have the same order.

The order of the operators of highest order in G' is twice the order of the operators of highest order in H_1. Hence the order of the operators of highest order in the group of order $2^{2n-1} h_1^{2n}$ in the given infinite system of indivisible groups derived from H_1 is 2^n times that of the operators of highest order in H_1. The number of these operators of highest order is easily seen to be

$$2 \cdot 2^{2^2-1} \cdot 2^{2^3-1} \cdots 2^{2^{n-1}-1} h_1^{1+2+2^2+\cdots+2^{n-1}} l = 2^{2^n-n-1} h_1^{2^n-1} l,$$

where l is the number of the operators of highest order contained in H_1.

Since each one of the largest abelian subgroups of a direct product must be the direct product of largest abelian subgroups of each of the factor groups, it results that the index of the former subgroups is the product of the indices of the latter. From this it follows directly that the infinite system of indivisible groups derived from H_1 in the given manner cannot involve more than one group which involves an abelian subgroup of index 2, and if there is one such group it must be H_1. If H_1 involves no abelian subgroup of index 2 the method employed above to prove that G' is indivisible yields also the result that G' involves only one subgroup of index 2 which is a square. Hence it results that there are never more than two groups including H_1, in the infinite system of indivisible groups derived from H_1 in the given manner, which have the property that they involve a subgroup of index 2 which is a square.

We proceed to prove now that if we derive, in the given manner, two infinite systems of indivisible groups from two given non-abelian indivisible groups H_1 and H_1', then no group of the one system can be simply isomorphic with a group of the other system unless one of these systems is contained entirely in the other. Suppose that H_1 and H_1' are two distinct groups which have the property that neither of them involves a subgroup of index 2 which is the square of a non-abelian indivisible group. The group G' derived from H_1 contains the square of H_1. We may suppose that H_1 is represented as a

regular substitution group and that this square is the product of two regular substitution groups on distinct sets of letters. These simply isomorphic regular substitution groups will again be represented by H_1 and H_2, and their product by G.

If this transitive substitution group G' on $2h_1$ letters contains a second subgroup of index 2 which is a square, this subgroup may be supposed to be the direct product of K_1, K_2 and it must involve half the operators of G. These operators are obtained by establishing an isomorphism between H_1 and H_2 with respect to a quotient group of order 2. The subgroup generated by them will be denoted by K. If K were the direct product of K_1 and a subgroup of index 2 contained in K_2, all the operators of K_1 would be commutative with operators of G' which are not contained in G. Hence K_1 would be simply isomorphic with H_1. It remains only to consider the case when K is obtained by establishing an isomorphism between K_1 and K_2 with respect to a quotient group of order 2. Hence there must be operators in G' which are not in G and are separately commutative with each of the operators of two subgroups of index 2 contained respectively in K_1 and K_2. As these two subgroups are simply isomorphic with corresponding subgroups of H_1 and H_2, since each of their substitutions, besides the identity, must involve all the letters of G', and as every substitution of the one is commutative with every substitution of the other, it results that K_1 and K_2 cannot differ from H_1 and H_2 unless each of these groups involves an abelian subgroup of index 2.

We proceed to prove that the groups K_1 and K_2 can be obtained by extending an abelian group of odd order by means of an operator of order 2 which transforms each operator of this abelian group into its inverse whenever these groups are not simply isomorphic with H_1 and H_2. In fact, it results from the above that K is a substitution group such that all of its substitutions besides the identity involve all the letters of G'. Hence K is a simple isomorphism between two regular groups. As all the substitutions of K_1 are commutative with K_2 it results that K_1 cannot involve any invariant substitution besides the identity, since such a substitution could not interchange systems of intransitivity of G, and hence it and K_2 would generate substitutions which would not involve all the letters of G'.

If a non-abelian group contains an abelian subgroup of index 2 but does not contain any invariant operator besides the identity, it is obtained by extending this abelian subgroup by an operator of order 2 which transforms each operator of this subgroup into its inverse and this subgroup is of odd order.* Hence we have established the following useful theorem: *The indivisible group obtained by extending the direct product of two simply isomorphic non-abelian indivisible groups by means of an operator of order 2 which transforms the*

*These Transactions, vol. 10 (1909), p. 472.

corresponding operators of these factor groups into each other cannot involve a second subgroup of index 2 which is such a direct product, unless each of the given non-abelian indivisible groups can be obtained by extending an abelian group of odd order by means of an operator of order 2 which transforms each operator of this abelian group into its inverse. From this theorem and the results proved above we deduce the theorem: If H_1 and H_1' are two non-abelian indivisible groups which do not involve a subgroup of index 2 which is the direct product of two simply isomorphic non-abelian indivisible groups, then there is no common group in the two infinite systems of indivisible groups, derived from H_1 and H_1' by the process of forming successively the direct product and extending this product by means of an operator of order 2, which transforms the corresponding operators of this direct product into each other.

It may be observed that when H_1 can be obtained by extending an abelian group of odd order by means of an operator of order 2 which transforms each operator of this abelian group into its inverse then G' will always contain more than one subgroup of index 2 which is the square of a group. These subgroups are, however, simply isomorphic, according to the proof given above, and hence the entire system is always independent of the choice of its first possible indivisible non-abelian group whenever this first group can be chosen in more than one way. In what follows we shall consider several infinite systems of indivisible groups which have elementary defining equations and which seem to be especially important. The infinite systems of such groups which may be derived from any given non-abelian indivisible group by the general methods given above will not be considered in the remaining sections of this article.

§ 4. *Several infinite systems of known groups which are indivisible.*

One of the most important systems of non-abelian groups is that composed of the dihedral groups. Since the group of inner isomorphisms of a dihedral group G is dihedral, it results from the preceding section that such a group cannot be the direct product of two non-abelian groups, as its group of inner isomorphisms involves a cyclic subgroup of prime index. It results therefore, that either G is indivisible or it is the direct product of a subgroup of its central and another subgroup. As the order of this central cannot exceed 2 unless G is the four-group and as the operator of order 2 in the central is a commutator whenever the order of the dihedral group is divisible by 8, it results that *a necessary and sufficient condition that the dihedral group be a direct product is that its order is divisible by* 4 *but not by* 8. If its order exceeds 4 and is divisible by 4 but not by 8, it is evident that G is the direct product of the group of order 2 and of the dihedral group whose order is half the order of G. The system of indivisible dihedral groups is therefore composed of one and only

one group of every even order which is not the product of 4 and some odd number. This system will hereafter be denoted by S_4. The first groups of S_2 and S_4 are identical but otherwise the system S_4 has clearly no group in common with any of the systems S_1, S_2, S_3.

Closely related to the system of indivisible groups S_4 is the system of dicyclic groups. A dicyclic group may be defined as a group that can be generated by two non-commutative operators of order 4 whose product generates their common square.* There is one and only one such group of every order which exceeds 4 and is divisible by 4. As the group of inner isomorphisms of a dicyclic group is dihedral and as the invariant operator of order two in such a group is generated by each one of its operators of order 4, it results from the theorem quoted in the preceding paragraph that a dicyclic group is necessarily indivisible. We shall denote the infinite system of these groups by S_5 and it is clear that S_5 does not have any group in common with any of the systems $S_1, \cdots, S_4$. The first group of S_5 is the quaternion group. In the following section we shall discuss a category of groups which includes S_4 and S_5.

It is well known that there is one and only one non-abelian group of order p^m, p being a prime and $m > 3$, which involves p cyclic subgroups of order p^{m-1}. This group exists also when $m = 3$ and $p > 2$. As the central of this group is cyclic it must be indivisible and hence all these groups constitute an infinite system of indivisible groups, which will be denoted by S_6. It is clear that S_6 does not have any group in common with any of the systems $S_1, \cdots, S_5$. A closely related infinite system of indivisible group is constituted by the groups of order 2^n which involve three subgroups of order 2^{n-1} which are dihedral, dicyclic and cyclic respectively. There is one and only one such group for every value of $n > 3$, and the centrals of these groups are cyclic. This infinite system of indivisible groups will be denoted by S_7 and it is clear that S_7 does not have any group in common with any of the systems $S_1, \cdots, S_6$.

It was observed in the preceding section that the holomorph of any cyclic group is the direct product of the holomorphs of its Sylow subgroups, and that the latter holomorphs are always indivisible. As the order of the holomorph of a cyclic group of order p^σ is $p^{2\sigma-1}(p-1)$ it is easy to see that the orders of the holomorphs of two such cyclic groups cannot be equal unless these cyclic groups are identical. In fact, an equation of the form $p^{2\sigma-1}(p-1) = q^{2\beta-1}(q-1)$ where p and q are primes is impossible, since the larger of these two primes could not divide the member of the equation in which it does not explicitly appear. Hence the holomorphs of the indivisible cyclic groups constitute an infinite system of indivisible groups which is such that no two of its group have the same order. This system will be

* Another definition of dicyclic groups is given by HILTON, *Finite Groups*, 1908, p. 150.

denoted by S_8, and it is composed of all the indivisible holomorphs of cyclic groups.

We proceed to prove that the holomorphs of the abelian groups of order p^α, p being any prime, and of type $(1, 1, 1, \cdots)$ constitute an infinite system of indivisible groups which is such that no two of its groups are of the same order. It is well known that the order of the holomorph of the given group is $p^\alpha (p^\alpha - 1)(p^\alpha - p) \cdots (p^\alpha - p^{\alpha-1})$, and hence we have first to prove that if p and q are any two different primes it is impossible to find values for α and β such that the equation

$$p^\alpha (p^\alpha - 1)(p^\alpha - p) \cdots (p^\alpha - p^{\alpha-1}) = q^\beta (q^\beta - 1)(q^\beta - q) \cdots (q^\beta - q^{\beta-1})$$

is satisfied. To prove this it is only necessary to observe that we may write this equation in the form

$$p^{\alpha_1} r = q^{\beta_1} s,$$

where $r < p^{\alpha_1}$ and $s < q^{\beta_1}$. Moreover, if this equation were true $r = l_1 q^{\beta_1}$ and $s = l_2 p^{\alpha_1}$, where l_1 and l_2 are natural numbers. Hence we have the two contradictory relations

$$rs < p^{\alpha_1} q^{\beta_1} \qquad \text{and} \qquad rs = l_1 l_2 p^{\alpha_1} q^{\beta_1}.$$

As the assumption that the equation

$$p^\alpha (p^\alpha - 1)(p^\alpha - p) \cdots (p^\alpha - p^{\alpha-1}) = q^\beta (q^\beta - 1)(q^\beta - q) \cdots (q^\beta - q^{\beta-1})$$

could be satisfied led to a contradiction we have established the theorem: *The orders of the holomorphs of two abelian groups each of which involves only operators of the same prime order, besides the identity, cannot be equal unless these abelian groups are identical.*

From this theorem and from the preceding section, it results directly that the holomorphs of the abelian groups whose orders are powers of a single prime and whose type is $(1, 1, 1, \cdots)$ constitute an infinite system of indivisible groups which is such that no two of its groups have the same order. This system will be denoted by S_9, and it is composed of all the indivisible holomorphs of the abelian groups which involve no operator whose order is the square of a prime number.

Another interesting system of indivisible groups is generated by n operators which are such that each one of them transforms each of the remaining $n - 1$ into its inverse. If we assume that at least one of these n operators has an order which exceeds 2 then all of them must be of order 4 and every pair of them must generate the quaternion group. For simplicity of statement we shall assume this and we shall also assume that none of these n operators is in the group generated by the remaining $n - 1$. Hence it results that they

generate a group of order 2^{n+1}, and it is known that there is one and only one group of order 2^α, $\alpha > 2$, which is generated by such a set of operators.* The first one of these groups is the quaternion group, which is indivisible, while the second is the Hamiltonian group of order 16 and hence it is divisible. We proceed to determine a necessary and sufficient condition that such a group G is divisible.

Let the n generating operators, which have the property that each one of them is transformed into its inverse by each of the others, be represented by $S_1, S_2, \cdots, S_n$. The subgroup generated by the first $\alpha < n$ of these contains one and only one subgroup of order 2^α which does not involve any one of the generating operators $S_1, S_2, \cdots, S_\alpha$. Each of the operators of this subgroup of order 2^α is commutative with every one of the operators $S_{\alpha+1}, \cdots, S_n$. This subgroup contains two and only two operators which are commutative with each of the operators $S_1, \cdots, S^\alpha$. These two operators are the commutators of G. It is known that the central of G is of order 2 when n is even and that this central is of order 4 when n is odd.† A necessary and sufficient condition that this central be the four-group is that $n + 1$ is a multiple of 4.

According to the preceding section it results therefore that G cannot be divisible unless $n \equiv 3 \pmod 4$. When n satisfies this condition the central of G is non-cyclic and G involves a subgroup of half its own order which must be indivisible, in accord with what has just been proved. As the central of this subgroup is of order 2, since its order is 2^β, β being odd, it follows that G involves an invariant operator of order 2 which is not contained in this subgroup and hence G is the direct product of the group of order 2 and an indivisible group whenever $n \equiv 3 \pmod 4$. This completes a proof of the following theorem: *There is one and only one indivisible group of order* 2^α, $\alpha \not\equiv 0 \pmod 4$, *which can be generated by a set of operators satisfying the condition that each one transforms each of the others into its inverse.* When $\alpha \equiv 0 \pmod 4$ there is again one and only one non-abelian group which can be generated by such a set of operators, but it is the direct product of the group of order 2 and an indivisible group. The system of indivisible groups described in the italicized theorem of this paragraph will be represented by S_{10}.

§ 5. *Groups which can be generated by two operators whose common square is generated by their product.*

Let S_1, S_2 be any two non-commutative operators which satisfy the conditions

$$S_1^2 = S_2^2 = (S_1 S_2)^n.$$

* Bulletin of the American Mathematical Society, vol. 18 (1912), p. 329.

† Ibidem.

It is evident that every dihedral group, except the four-group, and that every dicyclic group can be generated by two such operators. Some of the properties of all the possible groups which can be generated by S_1, S_2 when these operators are restricted only by the condition $S_1^2 = S_2^2$ have been determined.* In the present section we aim to determine additional properties of the special category of these groups which can be generated by S_1, S_2 when the second condition stated above is added.

As $(S_1 S_2)^n = (S_1 S_2^{-1})^n S_2^{2n} = S_1^2$ it results that S_1, S_2 must satisfy the equation

$$(S_1 S_2^{-1})^n = S^{2(1-n)}.$$

From the fact that $(S_1 S_2^{-1})^n$ is both invariant under the group G generated by S_1, S_2 and is also transformed into its inverse by S_1,† it follows that the order of $(S_1 S_2^{-1})^n$ is either 1 or 2. Since n and $1 - n$ are relatively prime, the order of $S_1 S_2^{-1}$ is not divisible by any odd prime number which divides the order of S_1. This fact may be used to prove that the two operators $S_1 S_2^{-1}$ and S_1^2 must always generate a cyclic group whose order is one half the order of G.

To prove this theorem it remains only to observe that the Sylow subgroup of order 2^m in the abelian group generated by $S_1 S_2^{-1}$ and S_1^2 is cyclic. As this abelian group involves $S_1 S_2$ and is generated by $S_1 S_2$ and $S_1 S_2^{-1}$, it results that the Sylow subgroup in question is always generated by $S_1 S_2$. In fact, the cyclic group generated by $S_1 S_2$ involves both S_1^2 and $S_1 S_2^{-1}$ since the former is generated, by hypothesis, by $S_1 S_2$. Hence the following theorem has been established. *If a group is generated by two operators whose common square is generated by their product, then it contains a cyclic subgroup of index two and this cyclic subgroup is generated by this product.*

Suppose that G is a non-abelian indivisible group which contains a cyclic subgroup H of index 2. It is clear that H cannot involve any invariant operators under G whose orders are prime numbers, for if it should involve any such operator the corresponding Sylow subgroup would be composed of invariant operators under G, and hence G would be a direct product, which is contrary to the hypothesis. Hence the invariant operators of G must have orders of the form 2^α whenever G is indivisible and contains a cyclic subgroup of index 2. Moreover, every operator of odd order contained in G is transformed into its inverse by all the operators of G which are not contained in H. Hence G can be constructed by establishing an isomorphism, with regard to a quotient group of order 2, between a dihedral group whose order is twice an odd number and a group of order 2^m which involves operators of order 2^{m-1}. If the latter group were either dihedral or dicyclic G would be dihedral or dicyclic, and hence we do not need to consider these two cases.

* MILLER: Archiv der Mathematik und Physik, ser. 3, vol. 9 (1905), p. 6.
† Ibidem.

If this group of order 2^m belongs to one of the systems S_6 or S_7 it results from section 2 that G is indivisible, since the Sylow subgroups of order 2^m are indivisible and each of them involves operators which are non-commutative with each of the operators of odd order contained in G. Hence we obtain in this way two additional systems of indivisible groups which are such that no two groups of the same system have the same order. The system obtained by selecting the Sylow subgroup of order 2^m from S_6 will be denoted by S_{11}, while the system obtained by selecting this Sylow subgroup from S_7 will be represented by S_{12}. None of the groups of the systems S_{11} and S_{12} can be generated by two operators whose common square is generated by their product, since the given Sylow subgroups of order 2^m cannot be generated in this way. We shall see that S_{11} and S_{12} are composed of all the indivisible groups which involve a cyclic subgroup of index 2 but cannot be generated by two such operators.

The only indivisible groups which involve a cyclic subgroup of index 2 and have not yet been considered are those in which the given Sylow subgroup of order 2^m is cyclic and $m > 2$. There is one and only one such group of every order which is divisible by 8 but is not a power of 2, since such an isomorphism can be established for every dihedral group and every cyclic group of order 2^m. All of these groups are indivisible since the Sylow subgroups of order 2^m are indivisible and each of them involves operators which are non-commutative with every operator of odd order contained in G. This infinite system of indivisible groups will be denoted by S_{13}, and we may express a part of the preceding results in the form of the following theorem: *An indivisible group which can be generated by two operators whose common square is generated by their product belongs to one of the four systems*: S_1, S_4, S_5 *and* S_{13}. The last three of these systems are composed of such indivisible groups while the first includes also other groups.

§ 6. *List of infinite systems of indivisible groups.*

For convenience we give below a list of the infinite systems of indivisible groups which considered individually have been in what precedes. Each of these systems is composed of indivisible groups which are such that no two groups of the same system have the same order. A few groups of low orders belong to more than one of these systems but the groups of higher order are all distinct. It is hoped that these symbols may serve as a convenient means of reference.

Symbol.	*Description.*
S_1	All the cyclic groups whose orders are powers of a single prime number.
S_2	The abstract groups which are simply isomorphic with the symmetric substitution groups.

S_3 The abstract groups which are simply isomorphic with the alternating substitution groups.

S_4 All dihedral groups except those whose orders are the product of 4 and some odd number.

S_5 All dicyclic groups.

S_6 The non-abelian groups of order p^m, p being a prime, which contain exactly p cyclic subgroups of order p^{m-1}.

S_7 The groups of order 2^n involving three subgroups of order 2^{n-1} which are respectively cyclic, dicyclic and dihedral.

S_8 The holomorphs of all the cyclic groups whose orders are powers of a single prime number.

S_9 The holomorphs of the abelian groups of order p^m and of type $(1, 1, \cdots)$.

S_{10} The non-abelian groups of order 2^α, $\alpha \not\equiv 0 \pmod 4$, which can be generated by a set of operators each of which transforms each of the others into its inverse.

S_{11}–S_{13} The groups containing a cyclic subgroup of index 2 and formed by an isomorphism between a dihedral group whose order is the double of an odd number and a group of order 2^m contained in the systems S_6, S_7 and S_1 respectively.

IMPROPER MULTIPLE INTEGRALS OVER ITERABLE FIELDS*

BY

J. K. LAMOND

Introduction.

LYTLE † has shown that the fundamental relation

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f,$$

where the integrals are proper integrals, holds for a class of fields which he calls iterable (defined later in § 1). In his *Lectures*,‡ vol. II, Professor PIERPONT has given a new definition of an improper multiple integral (definition given later in § 1). Considering functions defined, in general, over iterable fields and using Professor PIERPONT's definition, I have shown that, under certain conditions, the above fundamental inequalities hold for a very general class of functions, *some of which may have points of infinite discontinuity at an everywhere dense set in* $\mathfrak{A}$, *the field of integration.*

For simplicity the proofs are given for functions of two independent variables. The methods of proof, however, are perfectly general and may be extended readily to any number of independent variables.

§ 1. *Preliminary Definitions. Uniform and Regular Convergence.*

Let $f(x_1, \cdots, x_n)$ be a function, defined over a limited field $\mathfrak{A}$, and let it have infinite discontinuities at the points of a set $\mathfrak{I}$, which may or may not be of zero content. The following is PIERPONT's definition of the improper multiple integral of $f(x_1, \cdots, x_n)$ over $\mathfrak{A}$.

Let $\alpha \geqq 0$ and $\beta \geqq 0$ be two numbers, chosen at pleasure. Let $\mathfrak{A}_{\alpha\beta}$ be those points of $\mathfrak{A}$ at which

$$-\alpha \leqq f(x_1, \cdots, x_n) \leqq \beta$$

* Presented to the Society, April 27, 1912.

† *Proper Multiple Integrals over Iterable Fields*, Transactions of the American Mathematical Society, vol. 11 (1910), p. 25.

‡ *Lectures on the Theory of Functions of Real Variables*, Ginn & Co. In two volumes. These will be referred to as *Lectures*.

and let $\mathfrak{A}'_{\alpha\beta}$ be the remaining points of $\mathfrak{A}$. If

$$\lim_{\alpha,\,\beta=\infty} \overline{\int}_{\mathfrak{A}_{\alpha\beta}} f$$

exists, finite or infinite, it is defined to be the upper integral of $f(x_1, \cdots, x_n)$ over $\mathfrak{A}$ and is written

$$\overline{\int}_{\mathfrak{A}} f. \tag{1}$$

The lower integral of $f(x_1, \cdots, x_n)$ is similarly defined and written. When the upper and lower integrals of $f(x_1, \cdots, x_n)$ over $\mathfrak{A}$ are equal we denote their common values by

$$\int_{\mathfrak{A}} f$$

and say that $f(x_1, \cdots, x_n)$ is integrable in $\mathfrak{A}$.

Let now $\mathfrak{A}$ be a limited point aggregate in two dimensions. Let $\mathfrak{B}$ be the projection of $\mathfrak{A}$ on the X axis and through each point x of $\mathfrak{B}$ let an ordinate be erected. Each ordinate cuts a section $\mathfrak{C}(x)$ out of $\mathfrak{A}$. When no confusion can arise I denote this section simply by $\mathfrak{C}$. It is convenient to use the notation $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$.

For the definition of the upper content of a set of points see *Lectures*, vol. I. The notation there employed will be used here.

According to LYTLE'S definition* if the integral of the upper content of $\mathfrak{C}$ over $\mathfrak{B}$ exists and is equal to the upper content of $\mathfrak{A}$, that is, if

$$\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \bar{\mathfrak{A}},$$

then $\mathfrak{A}$ is said to be iterable, modulo $\mathfrak{B}$.

Let $\rho \geqq 0$ and $\sigma \geqq 0$ be two numbers chosen at pleasure. Let $\mathfrak{B}_{\rho\sigma}$ be those points of $\mathfrak{B}$ at which

$$-\rho \leqq \overline{\int}_{\mathfrak{C}} f \leqq \sigma. \tag{2}$$

If

$$\lim_{\rho,\,\sigma=\infty} \overline{\int}_{\mathfrak{B}_{\rho\sigma}} \overline{\int}_{\mathfrak{C}} f$$

exists, finite or infinite, it is defined to be the upper integral of the integral in (2) over $\mathfrak{B}$, and is written

$$\overline{\int}_{\mathfrak{A}} \overline{\int}_{\mathfrak{C}} f.$$

* Loc. cit.

The lower, lower iterated integral, and the iterated integral are similarly defined.

It is evident that f may be defined at the points of $\mathfrak{A}$ and yet the integral in (2) may not be defined at some or at any of the points of $\mathfrak{B}$. In this paper I shall restrict myself to the consideration of functions such that the integral in (2) shall exist and be finite, although not necessarily limited, except possibly at the points of a discrete set, at which points we may assign an arbitrary value to this integral without affecting the value of the iterated integral (see *Lectures* vol. II, § 60).

In what follows if an upper integral, a lower integral, or an integral is said to exist or to converge, I mean that it exists and is finite unless otherwise stated.

Let us suppose that the integral in (2) exists for each point x in $\mathfrak{B}$. Then if

$$\epsilon > 0, \quad \alpha_0, \ \beta_0, \quad \left| \bar{\int}_{\mathfrak{G}_{\alpha\beta}} f - \bar{\int}_{\mathfrak{G}} f \right| < \epsilon, \quad \alpha \geqq \alpha_0, \ \beta \geqq \beta_0,^*$$

and for each and every x and $\mathfrak{B}$, we say that the integral in (2) *converges uniformly in* $\mathfrak{B}$.

Let $\mathfrak{d}$ be that partial set of $\mathfrak{B}$ such that the integral in (2) converges uniformly in $\mathfrak{B} - \mathfrak{d}$ and converges for each x in $\mathfrak{d}$ except possibly at the points of a discrete set. Let $\mathfrak{d}_{\rho\sigma}$ be those points of $\mathfrak{d}$ at which relation (2) holds. If $\mathfrak{d}_{\rho\sigma}$ is discrete for each $\rho \geqq 0$, $\sigma \geqq 0$, and in addition $\bar{\mathfrak{B}}_{\rho\sigma} = \bar{\mathfrak{B}}$ for each $\rho \geqq \rho_0$, $\sigma \geqq \sigma_0$, ρ_0, σ_0 sufficiently large, we say that the integral in (2) *converges regularly in* $\mathfrak{B}$. If $\mathfrak{d}_{\rho\sigma}$ is discrete and $\bar{\mathfrak{B}}_{\rho\sigma} < \bar{\mathfrak{B}}$ for each $\rho \geqq 0$, $\sigma \geqq 0$, we say that the integral in (2) *converges semi-regularly in* $\mathfrak{B}$. If the integral in (2) converges regularly, or semi-regularly, in $\mathfrak{B}$, it follows at once, from *Lectures*, vol. II, § 360, that the set $\mathfrak{d}$ is of measure zero.

Similar definitions may be given using lower integrals or integrals.

§ 2. *Improper Integrals.*

THEOREM 1. *Let*

$$\bar{\int}_{\mathfrak{G}} f \tag{1}$$

converge uniformly in $\mathfrak{B}$. *Then* (1) *is limited in* $\mathfrak{B}$ *and the upper integral of* (1) *over* $\mathfrak{B}$ *exists.*

For by the definition of uniform convergence, choosing $\epsilon > 0$, small at

*This set of inequalities is written in accordance with Pierpont's notations. It means; for every positive ϵ there exists a pair of numbers, α_0 and β_0, such that

$$\left| \bar{\int}_{\mathfrak{G}_{\alpha\beta}} f - \bar{\int}_{\mathfrak{G}} f \right| < \epsilon$$

for any $\alpha \geqq \alpha_0$, and for any $\beta \geqq \beta_0$.

pleasure, there exists a fixed pair of numbers α, β such that

$$\left|\bar{\int}_{\mathfrak{C}} f\right| < \left|\bar{\int}_{\mathfrak{C}_{\alpha\beta}} f\right| + \epsilon$$

for each and every x in $\mathfrak{B}$. But

$$\left|\bar{\int}_{\mathfrak{C}_{\alpha\beta}} f\right| \leqq \mu \bar{B},$$

where B is the projection of $\mathfrak{A}$ on the Y axis and $\mu = \text{Max}(\alpha, \beta)$. Hence

$$\left|\bar{\int}_{\mathfrak{C}} f\right| < \mu \bar{B} + \epsilon < M$$

for each and every x in $\mathfrak{B}$.

THEOREM 2. *Let the upper integral of f over $\mathfrak{C}$ converge regularly, or semi-regularly, in $\mathfrak{B}$. Then*

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f$$

exists.

Let $F = 0$ on sections corresponding to $\mathfrak{d}$,

$= f$ on sections corresponding to $\mathfrak{B} - \mathfrak{d}$.

Then for any ρ, σ

$$\bar{\int}_{\mathfrak{B}_{\rho\sigma}} \bar{\int}_{\mathfrak{C}} f = \bar{\int}_{\mathfrak{B}_{\rho\sigma}} \bar{\int}_{\mathfrak{C}} F, \tag{1}$$

since for points of $\mathfrak{B}_{\rho\sigma}$ the upper integral of f over $\mathfrak{C}$ differs from the upper integral of F over $\mathfrak{C}$ at only a discrete set $\mathfrak{d}_{\rho\sigma}$. Let now $\rho, \sigma \doteq \infty$. The upper integral of F over $\mathfrak{C}$ converges uniformly in $\mathfrak{B}$. Hence, by Theorem 1, the limit on the right of (1) exists. Hence the limit on the left exists.

THEOREM 3. *Let f be limited in $\mathfrak{A}$ and let $\mathfrak{A}_1$ be a partial aggregate of $\mathfrak{A}$. Let, 1°, $f = 0$ in $\mathfrak{A} - \mathfrak{A}_1$, or, 2°, let $\bar{\mathfrak{A}}_1 = \bar{\mathfrak{A}}$. In either case*

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{A}_1} f, \qquad \bar{\int}_{\mathfrak{A}_1} f \leqq \bar{\int}_{\mathfrak{A}} f. \tag{1}$$

Effect a cubical division of space and let the cells containing points of both $\mathfrak{A}$ and $\mathfrak{A}_1$ be $d_1, \cdots, d_n$. Let the remaining cells which contain points of $\mathfrak{A}$ be $d'_1, \cdots, d'_m$. If condition 1° is fulfilled, $f = 0$ in cells d'_i. If condition 2° is fulfilled, $\Sigma d'_i$ is small at pleasure. In any cell d_i, if either condition 1° or condition 2° is fulfilled, Max f at points of $\mathfrak{A}_1 \leqq$ Max f at points of $\mathfrak{A}$ and Min f at points of $\mathfrak{A} \leqq$ Min f at points of $\mathfrak{A}_1$. Hence the inequalities (1) follow.

THEOREM 4. *Let the upper integral of f over $\mathfrak{C}$ converge regularly in $\mathfrak{B}$. Then*

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \lim_{\alpha,\, \beta = \infty} \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}_{\alpha\beta}} f. \tag{1}$$

By the definition of regular convergence, and Theorem 1, there exists a pair of numbers ρ_0, σ_0 such that

$$\bar{\mathfrak{B}}_{\rho\sigma} = \bar{\mathfrak{B}}, \tag{2}$$

$$\mathfrak{B} - \mathfrak{b} = \mathfrak{B}_{\rho\sigma} - \mathfrak{b}_{\rho\sigma}, \quad \text{for} \quad \rho \geqq \rho_0, \quad \sigma \geqq \sigma_0. \tag{3}$$

Since $\mathfrak{b}_{\rho\sigma}$ is discrete, it follows from (3) that

$$\bar{\int}_{\mathfrak{B}-\mathfrak{b}} \bar{\int}_{\mathfrak{C}} f = \bar{\int}_{\mathfrak{B}_{\rho\sigma}} \bar{\int}_{\mathfrak{C}} f.$$

Passing to the limit $\rho, \sigma = \infty$ in this relation we have, by Theorem 2,

$$\bar{\int}_{\mathfrak{B}-\mathfrak{b}} \bar{\int}_{\mathfrak{C}} f = \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f. \tag{4}$$

Relations (2) and (3) give $\bar{\mathfrak{B}} = \text{Upper Cont.}\,(\mathfrak{B} - \mathfrak{b})$. Hence, by Theorem 3, for any α, β, arbitrary but fixed,

$$\bar{\int}_{\mathfrak{B}-\mathfrak{b}} \bar{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}_{\alpha\beta}} f. \tag{5}$$

By *Lectures*, vol. II, § 150,

$$\lim_{\alpha,\beta=\infty} \bar{\int}_{\mathfrak{B}-\mathfrak{b}} \bar{\int}_{\mathfrak{C}_{\alpha\beta}} f = \bar{\int}_{\mathfrak{B}-\mathfrak{b}} \lim_{\alpha,\beta=\infty} \bar{\int}_{\mathfrak{C}_{\alpha\beta}} f = \bar{\int}_{\mathfrak{B}-\mathfrak{b}} \int_{\mathfrak{C}} f.$$

This with the limit of (5) gives

$$\bar{\int}_{\mathfrak{B}-\mathfrak{b}} \bar{\int}_{\mathfrak{C}} f \leqq \lim_{\alpha,\beta=\infty} \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}_{\alpha\beta}} f.$$

This relation with (4) gives (1).

COROLLARY 1. *Let $f \geqq 0$ and let the upper integral of f over $\mathfrak{C}$ converge semi-regularly in $\mathfrak{B}$. Then*

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \lim_{\alpha,\beta=\infty} \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}_{\alpha\beta}} f.$$

Since the upper integral of f over $\mathfrak{C}$ converges semi-regularly in $\mathfrak{B}$, we have relation (3) and hence equation (4). Since $f \geqq 0$, the inequality (5) is true. The limit of this with (4) gives the corollary.

In connection with relation (4) it is well to notice that if the upper integral of f over $\mathfrak{C}$ converges regularly in $\mathfrak{B}$, we may disregard the set $\mathfrak{b}$, of measure zero, in computing

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f.$$

That the inequality sign is necessary in (1) is shown by the following example.

Example 1. Let $\mathfrak{A}$ be the unit square.

Let $f = n$ at the points* $x = m/n$, y rational, in $\mathfrak{A}$,
$= 2$ at the points $x = m/n$, y irrational, in $\mathfrak{A}$,
$= 1$ at the remaining points in $\mathfrak{A}$.

Let† $\beta \geqq 2$ be arbitrary but fixed. For x irrational

$$\overline{\int_{\mathfrak{C}}} f = 1, \qquad \overline{\int_{\mathfrak{C}_\beta}} f = 1,$$

For x rational

$$\overline{\int_{\mathfrak{C}}} f = n, \qquad \overline{\int_{\mathfrak{C}_\beta}} f = 2$$

except for a discrete set in the case of the latter integral. Hence

$$\overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} f = 1, \qquad \lim_{\beta=\infty} \overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}_\beta}} f = 2.$$

It is convenient to introduce an auxiliary function $\phi_{\alpha\beta}$ defined as follows: $\phi_{\alpha\beta}$ shall be equal to f at the points of $\mathfrak{A}_{\alpha\beta}$ and equal to zero at the points of $\mathfrak{A}'_{\alpha\beta}$.

THEOREM 5. *Let both the integrals*

$$\underline{\int_{\mathfrak{A}}} f$$

exist. Let $\bar{\mathfrak{I}} = 0$. *Then*

$$\lim_{\alpha,\,\beta=\infty} \overline{\int_{\mathfrak{A}}} \phi_{\alpha\beta} = \overline{\int_{\mathfrak{A}}} f. \tag{1}$$

Since $\bar{\mathfrak{I}} = 0$, $\bar{\mathfrak{A}}'_{\alpha\beta}$ may be made small at pleasure by taking α, β sufficiently large. Let $\mathfrak{A}'_{\alpha\beta}$ be enclosed in a set of cells $d_1, \cdots, d_n$ of norm d. Then

$$\epsilon > 0, \qquad \alpha_0, \beta_0, d_0, \qquad \Sigma d_i < \epsilon, \qquad \alpha \geqq \alpha_0, \quad \beta \geqq \beta_0, \quad d \leqq d_0.$$

Let these last quantities be arbitrary but fixed. Let $\mathfrak{A}_1$ be points of $\mathfrak{A}$ lying in cells d_i, and $\mathfrak{A}_2$ be the remaining points of $\mathfrak{A}$. Since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unmixed, by *Lectures*, vol. II, § 46,

$$\overline{\int_{\mathfrak{A}}} f = \overline{\int_{\mathfrak{A}_1}} f + \overline{\int_{\mathfrak{A}_2}} f, \qquad \overline{\int_{\mathfrak{A}}} \phi_{\alpha\beta} = \overline{\int_{\mathfrak{A}_1}} \phi_{\alpha\beta} + \overline{\int_{\mathfrak{A}_2}} \phi_{\alpha\beta}. \tag{2}$$

Since $f = \phi_{\alpha\beta}$ at points of $\mathfrak{A}_2$,

$$\overline{\int_{\mathfrak{A}_2}} f = \overline{\int_{\mathfrak{A}_2}} \phi_{\alpha\beta}.$$

* All fractions, here and elsewhere, are supposed to be irreducible.

† In the case of a function of one sign, the notation may be simplified by dropping either the α or the β as the case may be.

‡ For the significance of this system of inequalities cf. footnote on page 436.

Hence, subtracting the equations in relation (2) and using *Lectures*, vol. II, §§ 51 and 48, 3 we have

$$(3)\quad \left|\bar{\int}_{\mathfrak{A}} f - \bar{\int}_{\mathfrak{A}} \phi_{\alpha\beta}\right| = \left|\bar{\int}_{\mathfrak{A}_1} f - \bar{\int}_{\mathfrak{A}_1} \phi_{\alpha\beta}\right| \leqq \left|\bar{\int}_{\mathfrak{A}_1} f\right| + \left|\bar{\int}_{\mathfrak{A}_1} \phi_{\alpha\beta}\right| \leqq \bar{\int}_{\mathfrak{A}_1} |f| + \bar{\int}_{\mathfrak{A}_1} |\phi_{\alpha\beta}|.$$

In $\mathfrak{A}_1$, $|\phi_{\alpha\beta}|$ equals $|f|$ at some points and equals zero at the remaining points. From this fact and *Lectures*, vol. II, § 48, 3 and § 53

$$(4)\quad \bar{\int}_{\mathfrak{A}_1} |\phi_{\alpha\beta}| \leqq \bar{\int}_{\mathfrak{A}_1} |f| < \tfrac{1}{2}\eta$$

where η may be taken small at pleasure by taking ϵ sufficiently small. Relations (3) and (4) give

$$(5)\quad \left|\bar{\int}_{\mathfrak{A}} f - \int_{\mathfrak{A}} \phi_{\alpha\beta}\right| < \eta.$$

Now $\epsilon \doteq 0$ and $\alpha, \beta \doteq \infty$ when $\eta \doteq 0$. Hence we get (1) by passing to the limit $\eta = 0$ in (5).

Theorems similar to Theorems 1, 2, 4, and 5 are readily proved for lower integrals.

THEOREM 6. *Let* $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ *be iterable modulo* $\mathfrak{B}$. *Let both the integrals*

$$\underline{\int}_{\mathfrak{A}} f$$

exist. Let $\bar{\mathfrak{I}} = 0$. *Let both the integrals*

$$(1)\quad \bar{\int}_{\mathfrak{C}} f$$

converge uniformly or regularly in $\mathfrak{B}$. *Then*

$$(2)\quad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{A}} f.$$

Let α, β be arbitrary but fixed, and let $\phi_{\alpha\beta}$ be the limited auxiliary function already defined. In any section $\mathfrak{C}$, $\mathfrak{C}_{\alpha\beta} < \mathfrak{C}$ and $\phi_{\alpha\beta}$ equals f at points of $\mathfrak{C}_{\alpha\beta}$ and equals zero at points of $\mathfrak{C}'_{\alpha\beta}$. Hence it follows from Theorem 3 that

$$\underline{\int}_{\mathfrak{C}} \phi_{\alpha\beta} \leqq \underline{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \bar{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \bar{\int}_{\mathfrak{C}} \phi_{\alpha\beta},$$

and hence that

$$(3)\quad \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} \phi_{\alpha\beta} \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} \phi_{\alpha\beta}.$$

By Lytle's theorem 14 (loc. cit., p. 35) we have

$$\underline{\int}_{\mathfrak{A}} \phi_{\alpha\beta} \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} \phi_{\alpha\beta} \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} \phi_{\alpha\beta} \leqq \overline{\int}_{\mathfrak{A}} \phi_{\alpha\beta},$$

and this with (3) gives

$$\underline{\int}_{\mathfrak{A}} \phi_{\alpha\beta} \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \overline{\int}_{\mathfrak{A}} \phi_{\alpha\beta}. \tag{4}$$

Passing to the limit $\alpha, \beta = \infty$ in this last relation, and using Theorem 5 and *Lectures*, vol. II, § 150, in case that the integrals (1) converge uniformly in $\mathfrak{B}$, and Theorems 4 and 5, in case the integrals (1) converge regularly in $\mathfrak{B}$, we have*

$$\underline{\int}_{\mathfrak{A}} f = \lim_{\alpha,\beta=\infty} \underline{\int}_{\mathfrak{A}} \phi_{\alpha\beta} \leqq \lim_{\alpha,\beta=\infty} \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f$$

$$\leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \lim_{\alpha,\beta=\infty} \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \lim_{\alpha,\beta=\infty} \overline{\int}_{\mathfrak{A}} \phi_{\alpha\beta} = \overline{\int}_{\mathfrak{A}} f.$$

This gives relation (2).

COROLLARY 1. *Let the conditions of Theorem* 6 *be satisfied and in addition let f be integrable in* $\mathfrak{A}$. *Then*

$$\int_{\mathfrak{A}} f = \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f.$$

If f is also integrable in each $\mathfrak{C}$ *in* $\mathfrak{B}$

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f.$$

Let B be the projection of $\mathfrak{A}$ on the Y axis and C the corresponding section.

COROLLARY 2. *Let* $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C} = B \cdot C$ *be iterable modulis* $\mathfrak{B}$, B. *Let f be integrable in* $\mathfrak{A}$.

Let the integrals (1) *converge uniformly or regularly in* $\mathfrak{B}$ *and the integrals*

$$\underline{\int}_{c} f$$

converge uniformly or regularly in B. Then

$$\int_{\mathfrak{A}} f = \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} = \underline{\int}_{B} \underline{\int}_{C} f.$$

Example 2. Let $\mathfrak{A}$ be the points in a unit square whose abscissas are rational.

* In case the integrals (1) converge uniformly in $\mathfrak{B}$ the second and fourth inequality signs may be left out of this relation.

Let f be defined over $\mathfrak{A}$ as follows:

$$\begin{aligned} f &= n(-1)^n \text{ at the points } x = 1/n, & 0 \leqq y \leqq 1, \\ &= 1 \quad \text{at the points } x \neq 1/n, & y \text{ rational}, \\ &= 2 \quad \text{at the points } x \neq 1/n, & y \text{ irrational}. \end{aligned}$$

Then we have

$$\underline{\int}_{\mathfrak{A}} f = 1, \qquad \bar{\int}_{\mathfrak{A}} f = 2, \qquad \underline{\int}_{\mathfrak{C}} f = n(-1)^n$$

for $x = 1/n$. For x rational but not equal to $1/n$,

$$\underline{\int}_{\mathfrak{C}} f = 1, \qquad \bar{\int}_{\mathfrak{C}} f = 2.$$

Hence

$$\underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = 1, \qquad \int_{\mathfrak{B}} \int_{\mathfrak{C}} f = 2.$$

In connection with this example we may note the following points. The integrals over $\mathfrak{C}$ converge regularly in $\mathfrak{B}$; $\bar{\mathfrak{I}} = 0$; $\mathfrak{A}$ does not possess content; $\mathfrak{B}$ neither possesses content nor is complete; and the function f is limited in neither direction.

Example 3. Let $\mathfrak{A}$ be a unit square and let f be defined over $\mathfrak{A}$ as follows:

$$\begin{aligned} f &= n \text{ at points } x = m/n, & 0 \leqq y \leqq 1/n, \\ &= 1 \text{ elsewhere in } \mathfrak{A}. \end{aligned}$$

Then

$$\bar{\int}_{\mathfrak{A}} f = 1, \qquad \bar{\int}_{\mathfrak{C}} f = n \int_0^{1/n} dy + \int_{1/n}^1 dy = 2 - 1/n$$

for $x = m/n$. Elsewhere in $\mathfrak{B}$

$$\bar{\int}_{\mathfrak{C}} f = 1.$$

Hence

$$\underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = 1, \qquad \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f = 2.$$

We see that the fundamental relation breaks down. However

$$\underline{\int}_{\mathfrak{C}} f$$

do not converge either uniformly or regularly in $\mathfrak{B}$.

THEOREM 7. *Let f be of one sign or zero and defined over $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ which*

may or may not be iterable modulo $\mathfrak{B}$. *Let* $\bar{\mathfrak{I}} \geqq 0$. *Then, if* $f \geqq 0$ *and the upper integral of* f *over* $\mathfrak{A}$ *exists and the upper integral of* f *over* $\mathfrak{C}$ *converges uniformly, regularly, or semi-regularly in* $\mathfrak{B}$,

$$\int_{\mathfrak{B}}^{-} \int_{\mathfrak{C}}^{-} f \leqq \int_{\mathfrak{A}}^{-} f. \tag{1}$$

If $f \leqq 0$, *and the lower integral of* f *over* $\mathfrak{A}$ *exists and the lower integral of* f *over* $\mathfrak{C}$ *converges uniformly, regularly, or semi-regularly in* $\mathfrak{B}$, *then*

$$\int_{\mathfrak{A}}^{_} f \leqq \int_{\mathfrak{B}}^{_} \int_{\mathfrak{C}}^{_} f. \tag{2}$$

Let us prove (1); (2) is proved in a similar manner. Let ϕ_β be the auxiliary function already defined. As in the latter part of equation (3), Theorem 6,

$$\int_{\mathfrak{B}}^{-} \int_{\mathfrak{C}_\beta}^{-} f \leqq \int_{\mathfrak{B}}^{-} \int_{\mathfrak{C}}^{-} \phi_\beta. \tag{3}$$

By two theorems proved by Richardson *

$$\int_{\mathfrak{B}}^{-} \int_{\mathfrak{C}}^{-} \phi_\beta \leqq \int_{\mathfrak{A}}^{-} \phi_\beta = \int_{\mathfrak{A}_\beta}^{-} f.$$

This with relation (3) gives

$$\int_{\mathfrak{B}}^{-} \int_{\mathfrak{C}_\beta}^{-} f \leqq \int_{\mathfrak{A}_\beta}^{-} f.$$

Passing to the limit as in equation (4), Theorem 6, we have (1).

Example 4. Let f be a function defined over a set of points $\mathfrak{A}$, lying in a unit square, as follows:

$$\begin{aligned} f &= n \cdot q \text{ at the points } x = m/n, \quad y = 1/q, \\ &= n \text{ for } x \text{ irrational}, \quad y = m/n, \\ &= 1 \text{ for } x \text{ irrational}, \quad y \text{ irrational}. \end{aligned}$$

$$\int_{\mathfrak{A}}^{_} f = 1, \qquad \int_{\mathfrak{B}}^{_} \int_{\mathfrak{C}}^{-} f = 0, \qquad \int^{-} \int_{\mathfrak{C}}^{_} f = 1.$$

Here $\bar{\mathfrak{I}} > 0$. $\mathfrak{A}$ is not iterable since $\bar{\mathfrak{C}}(x)$ equals zero for x rational and equals 1 for x irrational. Hence

$$\int_{\mathfrak{B}} \bar{\mathfrak{C}}(x)$$

does not exist.

* *The Integration of a Sequence of Functions and Its Application to Iterated Integrals*, Transactions of the American Mathematical Society, vol. 9 (1908), p. 347.

THEOREM 8. *Let* $\mathfrak{A}_{\alpha\beta} = \mathfrak{B} \cdot \mathfrak{C}_{\alpha\beta}$ *be iterable modulo* $\mathfrak{B}$ *for any* $\alpha \geqq \alpha_0$, $\beta \geqq \beta_0$; α_0, β_0 *sufficiently large. Let* $\mathfrak{F} \geqq 0$. *Let both the integrals*

$$\underline{\int}_{\mathfrak{A}} f$$

exist. Let both the integrals

$$\underline{\int}_{\mathfrak{A}} f$$

converge uniformly or regularly in $\mathfrak{B}$. *Then*

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f. \tag{1}$$

For by Lytle's Theorem 14 (loc. cit., p. 35), for $a \geqq \alpha_0$, $\beta \geqq \beta_0$,

$$\underline{\int}_{\mathfrak{A}_{\alpha\beta}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}_{\alpha\beta}} f \leqq \overline{\int}_{\mathfrak{A}_{\alpha\beta}} f.$$

Passing to the limit $\alpha, \beta = \infty$ in this relation as in equation (4) of Theorem 6 we get (1).

Example 5. Let $\mathfrak{A}$ be the points of a unit square whose abscissas are rational and let f be defined over $\mathfrak{A}$ as follows:

$f = n$ at the points $x = m/2^n$, y rational,

$= 1$ at the points $x = m/2^n$, y irrational,

$= 2$ at the remaining points of $\mathfrak{A}$.

$$\underline{\int}_{\mathfrak{A}} f = 1, \quad \overline{\int}_{\mathfrak{A}} f = 2, \quad \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = 1, \quad \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f = 2.$$

WESLEYAN UNIVERSITY,
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ON A THEOREM OF FEJÉR'S AND AN ANALOGON TO GIBBS' PHENOMENON*

BY

T. H. GRONWALL

§ 1. *Introduction.*

Fejér† has obtained the following general result concerning the partial sums of a Fourier series:

Let $f(x)$ be a function, finite and integrable for $0 \leqq x \leqq 2\pi$; suppose that

$$|f(x)| \leqq M, \qquad 0 \leqq x \leqq 2\pi, \tag{1}$$

and that the Fourier coefficients of $f(x)$ satisfy the inequalities

$$|a_n| \leqq \frac{A}{n}, \qquad |b_n| \leqq \frac{B}{n}, \tag{2}$$

which is the case, for instance, when $f(x)$ is a function of limited fluctuation. Let

$$s_n(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx \tag{3}$$

be the $(n+1)$th partial sum of the Fourier series for $f(x)$, and

$$S_n(x) = \frac{1}{n+1}[s_0(x) + s_1(x) + \cdots + s_n(x)]; \tag{4}$$

then‡

$$|S_n(x)| \leqq M, \qquad 0 \leqq x \leqq 2\pi, \tag{5}$$

and from the identity

$$s_n(x) = S_n(x) + \frac{1}{n+1}\sum_{\nu=1}^{n} \nu\,(a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{6}$$

we conclude by the aid of (2) and (5) that

$$|s_n(x)| \leqq M + \frac{1}{n+1}\left(A\sum_{\nu=1}^{n}|\cos \nu x| + B\sum_{\nu=1}^{n}|\sin \nu x|\right), \tag{7}$$

* Presented to the Society (Chicago), April 5, 1912.

† L. FEJÉR, *Sur les sommes partielles de la série de Fourier*, Comptes Rendus, 23 mai, 1910, and *Sur les singularités de la série de Fourier des fonctions continues*, Annales de l'École Normale, ser. 3: vol. 28 (1911), pp. 63–103.

‡ L. FEJÉR, *Untersuchungen über trigonometrische Reihen*, Mathematische Annalen, vol. 58 (1904), pp. 51–69.

whence Fejér's inequality

$$|s_n(x)| \leqq M + \frac{n}{n+1}(A+B) < M + A + B.$$

In § 2 of the present paper, it is shown that the above inequality may be replaced by the following closer ones:

$$|s_n(x)| < M + \sqrt{A^2+B^2} \tag{8}$$

for any values of A and B, and

$$|s_n(x)| < M + \frac{1}{2}A + \frac{\sqrt{3}}{2}B \tag{9}$$

for $B \gtrless \sqrt{3}\,A$.

We then proceed to investigate the expressions occurring in (7):

$$\begin{aligned} U_n(x) &= \frac{1}{n+1}(|\cos x| + |\cos 2x| + \cdots + |\cos nx|), \\ V_n(x) &= \frac{1}{n+1}(|\sin x| + |\sin 2x| + \cdots + |\sin nx|). \end{aligned} \tag{10}$$

The first question demanding a solution is obviously that of the existence of $\lim_{n=\infty} U_n(x)$ and $\lim_{n=\infty} V_n(x)$. In § 3 it is shown that for $x = 0$

$$\lim_{n=\infty} U_n(0) = 1, \qquad \lim_{n=\infty} V_n(0) = 0;$$

that for $x = (p/q)\,\pi$, where $p\,(\neq 0)$ and q are integers and relative primes

$$\lim_{n=\infty} U_n\left(\frac{p}{q}\pi\right) = \frac{\sin\dfrac{\left[\dfrac{q}{2}\right]+\dfrac{1}{2}}{q}\pi}{q\sin\dfrac{\pi}{2q}} = \frac{1}{2q}\left(\cot\frac{\pi}{4q} + (-1)^{q+1}\tan\frac{\pi}{4q}\right)$$

$$= \begin{cases} \dfrac{1}{q}\cot\dfrac{\pi}{2q}, & q \text{ even}, \\ \dfrac{1}{q\sin\dfrac{\pi}{2q}}, & q \text{ odd}, \end{cases} \tag{11}$$

$$\lim_{n=\infty} V_n\left(\frac{p}{q}\pi\right) = \frac{1}{q}\cot\frac{\pi}{2q};$$

and finally that for $x : \pi$ irrational

$$\lim_{n=\infty} U_n(x) = \lim_{n=\infty} V_n(x) = \frac{2}{\pi}. \tag{12}$$

Thus the expressions $\lim_{n=\infty} U_n(x)$ and $\lim_{n=\infty} V_n(x)$ are simple examples of point-wise discontinuous functions of x, and it is interesting to note that

they arise quite naturally from the elementary problem in Fourier's series considered above, while most of the known examples of such functions have to be constructed in a more or less artificial way.

From (11) and (12) it is seen that for any fixed value of x

$$\begin{aligned} &\frac{1}{2} \overline{\overline{\leqslant}} \lim_{n=\infty} U_n(x) \overline{\overline{\leqslant}} 1, \\ &0 \overline{\overline{\leqslant}} \lim_{n=\infty} V_n(x) \overline{\overline{\leqslant}} \frac{2}{\pi}. \end{aligned} \tag{13}$$

Let x_n' be the value of x which makes $U_n(x)$ an absolute minimum (the absolute maximum is obviously $n/(n+1)$, for $x=0$), and x_n'' the value of x making $V_n(x)$ an absolute maximum (the absolute minimum $= 0$ being attained for $x = 0$); it will be shown in § 4 that

$$\lim_{n=\infty} U_n(x_n') = \frac{1}{2}, \tag{14}$$

and in § 5 that

$$V_n(x_n'') < \lim_{n=\infty} V_n(x_n'') = \sin z_0, \tag{15}$$

where z_0 is the smallest positive root of the equation

$$\tan \frac{z_0}{2} = z_0.$$

On account of $\sin z_0 > 2/\pi$, it is shown by (13), (14) and (15) that $V_n(x)$ presents an analogon to Gibbs' phenomenon, while there is none in the case of $U_n(x)$. Furthermore, $V_n(x_n'')$ increases monotonously with n, and, for $A = 0$, the inequality (9) may be replaced by

$$|s_n(x)| < M + B \sin z_0 < M + \frac{3}{4} B.$$

§ 2. *Demonstration of the relations* (8) *and* (9).

The identity

$$n(c_1^2 + c_2^2 + \cdots + c_n^2) = (c_1 + c_2 + \cdots + c_n)^2 + \frac{1}{2} \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} (c_\mu - c_\nu)^2$$

gives, when $c_1, \cdots, c_n$ are any real quantities,

$$\frac{c_1 + c_2 + \cdots + c_n}{n} \overline{\overline{\leqslant}} \sqrt{\frac{c_1^2 + c_2^2 + \cdots + c_n^2}{n}}, \tag{16}$$

and making $c_\nu = |\cos \nu x|$ and $c_\nu = |\sin \nu x|$ respectively, we obtain from (7)

$$|s_n(x)| \overline{\overline{\leqslant}} M + \frac{n}{n+1} \left(A \sqrt{\frac{\cos^2 x + \cos^2 2x + \cdots + \cos^2 nx}{n}} + B \sqrt{\frac{\sin^2 x + \sin^2 2x + \cdots + \sin^2 nx}{n}} \right),$$

or, writing

$$y = \sqrt{\frac{\sin^2 x + \sin^2 2x + \cdots + \sin^2 nx}{n}}, \tag{17}$$

$$|s_n(x)| \leqq M + \frac{n}{n+1}(A\sqrt{1-y^2} + By). \tag{18}$$

The expression $A\sqrt{1-y^2} + By$ becomes a maximum for

$$-\frac{Ay}{\sqrt{1-y^2}} + B = 0,$$

whence

$$y^2 = \frac{B^2}{A^2+B^2}, \qquad 1 - y^2 = \frac{A^2}{A^2+B^2},$$

$$A\sqrt{1-y^2} + By = \sqrt{A^2+B^2},$$

and by comparison with (18)

$$|s_n(x)| \leqq M + \frac{n}{n+1}\sqrt{A^2+B^2}, \tag{19}$$

from which (8) immediately follows. For $A = 0$ or $B = 0$, (8) coincides with Fejér's inequality.

We shall now proceed to prove (9) which, for $B \geqq \sqrt{3}A$, gives a closer limitation than either of the preceding ones. From

$$\sum_{\nu=1}^{n} \sin^2 \nu x = \frac{1}{2}\sum_{\nu=1}^{n}(1 - \cos 2\nu x) = \frac{n}{2} + \frac{1}{4} - \frac{\sin(2n+1)x}{4\sin x},$$

$$y^2 = \frac{1}{2} + \frac{1}{4n} - \frac{\sin(2n+1)x}{4n\sin x}. \tag{20}$$

As we obviously have

$$y^2(\pi + x) = y^2(\pi - x) = y^2(x),$$

we need consider the interval $0 \leqq x \leqq \pi/2$ only. First, making

$$x = \frac{2m\pi + z}{2n+1}, \qquad 0 \leqq m \leqq \left[\frac{n}{2}\right], \qquad 0 \leqq z \leqq \pi,$$

we find

$$y^2 = \frac{1}{2} + \frac{1}{4n} - \frac{\sin z}{4n \sin \dfrac{2m\pi + z}{2n+1}},$$

whence

$$y^2 \leqq \frac{1}{2} + \frac{1}{4n} \text{ for } \frac{2m\pi}{2n+1} \leqq x \leqq \frac{(2m+1)\pi}{2n+1}, \qquad 0 \leqq m \leqq \left[\frac{n}{2}\right]. \tag{21}$$

In the second place, making

$$x = \frac{(2m+1)\pi + z}{2n+1}, \qquad 0 \leqq m \leqq \left[\frac{n-1}{2}\right], \qquad 0 \leqq z \leqq \pi,$$

we find

$$y^2 = \frac{1}{2} + \frac{1}{4n} + \frac{\sin z}{4n \sin \dfrac{(2m+1)+z}{2n+1}}, \tag{22}$$

and to make this expression a maximum, z must be a root of the equation

$$\tan \frac{(2m+1)\pi + z}{2n+1} - \frac{1}{2n+1} \tan z = 0,$$

which obviously has a root in the interval $0 < z \leqq \pi/2$, but none in the interval $\pi/2 < z \leqq \pi$. Making $z = \pi/2 - \zeta$, the equation becomes

$$\tan \zeta - \frac{1}{2n+1} \cot \frac{(2m+1)\pi + \pi/2 - \zeta}{2n+1} = 0,$$

and as $0 < \zeta \leqq \pi/2$,

$$\zeta < \tan \zeta = \frac{1}{2n+1} \cot \frac{(2m+1)\pi + \frac{\pi}{2} - \zeta}{2n+1}$$

$$< \frac{1}{(2m+1)\pi + \pi/2 - \zeta} < \frac{1}{(2m+1)\pi},$$

whence

$$\frac{\pi}{2} - \frac{1}{(2m+1)\pi} < z \leqq \frac{\pi}{2}. \tag{23}$$

Now we first suppose $2m + 1 = n$; the equation determining ζ becomes

$$\tan \zeta - \frac{1}{2n+1} \tan \frac{\zeta}{2n+1} = 0$$

or

$$\zeta \tan \zeta - \frac{\zeta}{2n+1} \tan \frac{\zeta}{2n+1} = 0,$$

and as $x \tan x$ is a monotonously increasing function for $0 \leqq x \leqq \pi/2$, the above equation is satisfied only by $\zeta = 0$, whence $z = \pi/2$,

$$\frac{(2m+1)\pi + z}{2n+1} = \frac{\pi}{2}$$

and from (22)

$$y^2 = \frac{1}{2} + \frac{1}{4n} + \frac{1}{4n}$$

$$= \frac{1}{2} + \frac{1}{4n} + \frac{2n+1}{4n} \cdot \frac{1}{4m+3}. \tag{24}$$

In the remaining case, we have $1 \leqq 2m + 1 \leqq n - 1$, and for all integral values of m satisfying this inequality, the expression $(2m + 1)(n - 2m - 1)\pi$

attains its minimum value $= (n-1)\pi$ for $2m+1=1$ and $2m+1=n-1$; as we have

$$(n-1)\pi > \frac{2n}{\pi}$$

for $n \gtreqless 2$, we find

$$(2m+1)(n-2m-1)\pi > \frac{2n}{\pi},$$

or

$$\frac{\pi}{2} - \frac{1}{(2m+1)\pi} > \frac{2m+1}{2n}\pi.$$

By the aid of (23), this gives

$$z > \frac{2m+1}{2n}\pi$$

or

$$z > \frac{(2m+1)\pi + z}{2n+1},$$

whence, the expression $\sin z/z$ decreasing monotonously for z increasing from 0 to π,

$$\frac{\sin z}{z} < \frac{\sin\dfrac{(2m+1)\pi+z}{2n+1}}{\dfrac{(2m+1)\pi+z}{2n+1}}.$$

This inequality compared with (22) gives

$$y^2 < \frac{1}{2} + \frac{1}{4n} + \frac{2n+1}{4n}\cdot\frac{z}{(2m+1)\pi+z},$$

and as, for z limited by (23), the last fraction attains its maximum for $z=\pi/2$,

$$y^2 < \frac{1}{2} + \frac{1}{4n} + \frac{2n+1}{4n}\cdot\frac{1}{4m+3} \quad \text{for} \quad \frac{(2m+1)\pi}{2n+1} \gtreqless x \lesseqgtr \frac{(2m+2)\pi}{2n+1} \quad (n \gtreqless 2).$$

The comparison of this result with (21) and (24) finally gives

$$y^2 \gtreqless \frac{2n+1}{4n}\left(1+\frac{1}{4m+3}\right) \tag{25}$$

for

$$\frac{2m\pi}{2n+1} \gtreqless x \lesseqgtr \frac{(2m+2)\pi}{2n+1}, \qquad 0 \gtreqless m \lesseqgtr \left[\frac{n}{2}\right] \qquad (n \gtreqless 2),$$

whence

$$y^2 \gtreqless \frac{2n+1}{4n}\left(1+\frac{1}{3}\right) \lesseqgtr \frac{3}{4} \quad \text{for} \quad n \gtreqless 4.$$

Now, for $B \gtreqless \sqrt{3}A$, the expression $A\sqrt{1-y^2} + By$ attains its maximum

in the interval $0 \leqq y^2 \leqq \frac{3}{4}$ for $y^2 = \frac{3}{4}$, and (18) gives

$$|s_n(x)| \leqq M + \frac{n}{n+1}\left(\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right) \qquad (n \geqq 4),$$

whence (9) is proved for $n \geqq 4$; for $n = 1, 2, 3$ it is easily verified by numerical calculation.

§ 3. *Determination of* $\lim_{n=\infty} U_n(x)$ *and* $\lim_{n=\infty} V_n(x)$.

In order to obtain these limits, we have to consider three cases: 1°, $x = 0$; 2°, $x : \pi$ rational and $\neq 0$; 3°, $x : \pi$ irrational.

In the first case, we obviously have

$$U_n(0) = \frac{n}{n+1}, \qquad V_n(0) = 0,$$

whence

$$\lim_{n=\infty} U_n(0) = 1, \qquad \lim_{n=\infty} V_n(0) = 0.$$

In the second case, we have $x = \pi p/q$, where p and q are relative primes, and we may obviously suppose $q > 0$. As $|\cos \nu x|$ has the period π, we obtain

$$\sum_{\nu=1}^{n}\left|\cos\frac{\nu p}{q}\pi\right| = \sum_{\nu=1}^{q} + \sum_{\nu=q+1}^{2q} + \cdots + \sum_{\nu=[n/q]q-q+1}^{[n/q]q} + \sum_{\nu=[n/q]q+1}^{n}$$

$$= \left[\frac{n}{q}\right]\sum_{\nu=1}^{q}\left|\cos\frac{\nu p}{q}\pi\right| + \sum_{\nu=[n/q]q+1}^{n}\left|\cos\frac{\nu p}{q}\pi\right|.$$

We further have $[n/q] = (n/q) - \theta$, $0 \leqq \theta < 1$, and

$$\sum_{\nu=[n/q]q+1}^{n}\left|\cos\frac{\nu p}{q}\pi\right| \leqq \sum_{\nu=[n/q]q+1}^{n} 1 = n - \left[\frac{n}{q}\right]q = \theta q,$$

whence

$$\lim_{n=\infty} U_n\left(\frac{p}{q}\pi\right) = \frac{1}{q}\sum_{\nu=1}^{q}\left|\cos\frac{\nu p}{q}\pi\right|.$$

Now the numbers $p, 2p, \cdots, qp$ have the same residues modulo q (in a different order) as the numbers $1, 2, \cdots, q$, so that

$$\lim_{n=\infty} U_n\left(\frac{p}{q}\pi\right) = \frac{1}{q}\sum_{\nu=1}^{q}\left|\cos\frac{\nu p}{q}\pi\right| = \frac{1}{q}\sum_{\nu=1}^{q}\left|\cos\frac{\nu\pi}{q}\right|$$

$$= \frac{1}{q}\sum_{\nu=1}^{[q/2]}\cos\frac{\nu\pi}{q} - \frac{1}{q}\sum_{\nu=[q/2]+1}^{q}\cos\frac{\nu\pi}{q}$$

$$= \frac{1}{2q\sin\frac{\pi}{2q}}\left(\sin\frac{\left[\frac{q}{2}\right]+\frac{1}{2}}{q}\pi - \sin\frac{\pi}{2q}\right)$$

$$- \frac{1}{2q\sin\frac{\pi}{2q}}\left(\sin\frac{q+\frac{1}{2}}{q}\pi - \sin\frac{\left[\frac{q}{2}\right]+\frac{1}{2}}{q}\pi\right)$$

$$= \frac{\sin\frac{\left[\frac{q}{2}\right]+\frac{1}{2}}{q}\pi}{q\sin\frac{\pi}{2q}}$$

$$= \begin{cases} \frac{1}{q}\cot\frac{\pi}{2q}, & q \text{ even}, \\ \frac{1}{q\sin\frac{\pi}{2q}}, & q \text{ odd}, \end{cases}$$

and the last two expressions are obviously contained in

$$\frac{\cos^2\frac{\pi}{4q} + (-1)^{q+1}\sin^2\frac{\pi}{4q}}{q\sin\frac{\pi}{2q}} = \frac{1}{2q}\left(\cot\frac{\pi}{4q} + (-1)^{q+1}\tan\frac{\pi}{4q}\right).$$

In exactly the same way we find

$$\sum_{\nu=1}^{n}\left|\sin\frac{\nu p}{q}\pi\right| = \sum_{\nu=1}^{q} + \sum_{\nu=q+1}^{2q} + \cdots + \sum_{\nu=[n/q]q-q+1}^{[n/q]q} + \sum_{\nu=[n/q]q+1}^{n}$$

$$= \left[\frac{n}{q}\right]\sum_{\nu=1}^{q}\left|\sin\frac{\nu p}{q}\pi\right| + \sum_{\nu=[n/q]q+1}^{n}\left|\sin\frac{\nu p}{q}\pi\right|,$$

$$\lim_{n=\infty} V_n\left(\frac{p}{q}\pi\right) = \frac{1}{q}\sum_{\nu=1}^{q}\left|\sin\frac{\nu p}{q}\pi\right| = \frac{1}{q}\sum_{\nu=1}^{q}\left|\sin\frac{\nu\pi}{q}\right| = \frac{1}{q}\sum_{\nu=1}^{q}\sin\frac{\nu\pi}{q} = \frac{1}{q}\cot\frac{\pi}{2q}.$$

We finally consider the third case, where $x:\pi$ is irrational. In order to evaluate our expressions, we begin by developing the even function $|\cos x|$, which obviously satisfies Dirichlet's conditions for $0 \leqq x \leqq 2\pi$, in a Fourier series:

$$|\cos x| = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x,$$

where

$$a_\nu = \frac{1}{\pi}\int_0^{2\pi} |\cos x| \cos \nu x dx = \frac{1+(-1)^\nu}{\pi}\int_{-\pi/2}^{\pi/2} \cos x \cos \nu x dx,$$

whence, performing the integration,

$$|\cos x| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{\nu=1}^{\infty}(-1)^\nu\left(\frac{1}{2\nu+1} - \frac{1}{2\nu-1}\right)\cos 2\nu x, \tag{26}$$

and on account of the periodicity of both members, this formula is valid for all real values of x. Consequently, we have

$$|\cos x| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{\nu=1}^{N}(-1)^\nu\left(\frac{1}{2\nu+1} - \frac{1}{2\nu-1}\right)\cos 2\nu x + R_N, \tag{27}$$

where

$$|R_N| \lessgtr \frac{\pi}{2}\sum_{\nu=N+1}^{\infty}\left(\frac{1}{2\nu-1} - \frac{1}{2\nu+1}\right) = \frac{2}{\pi}\cdot\frac{1}{2N+1}. \tag{28}$$

By (27), we obtain

$$U_n(x) = \frac{2}{\pi}\cdot\frac{n}{n+1} + \frac{2}{\pi}\sum_{\nu=1}^{N}(-1)^\nu\left(\frac{1}{2\nu+1} - \frac{1}{2\nu-1}\right)\cdot\frac{1}{n+1}\sum_{\lambda=1}^{n}\cos 2\lambda\nu x + \epsilon_N, \tag{29}$$

where, on account of (28),

$$\begin{aligned}|\epsilon_N| &\lessgtr \frac{1}{n+1}\Big(|R_N(x)| + |R_N(2x)| + \cdots + |R_N(nx)|\Big) \\ &\lessgtr \frac{n}{n+1}\cdot\frac{2}{\pi}\cdot\frac{1}{2N+1} < \frac{2}{\pi}\cdot\frac{1}{2N+1}.\end{aligned} \tag{30}$$

Now, $x : \pi$ being irrational, none of the expressions $\sin \nu x$ $(\nu = 1, 2, \cdots, N)$ is equal to zero, so that

$$\lim_{n=\infty}\frac{1}{n+1}\sum_{\lambda=1}^{n}\cos 2\lambda\nu x = \lim_{n=\infty}\frac{1}{n+1}\frac{\sin n\nu x \cos(n+1)\nu x}{\sin \nu x} = 0,$$

and (29) and (30) consequently give

$$\limsup_{n=\infty} U_n(x) \lessgtr \frac{2}{\pi} + \frac{2}{\pi}\cdot\frac{1}{2N+1},$$

$$\liminf_{n=\infty} U_n(x) \gtrless \frac{2}{\pi} - \frac{2}{\pi}\cdot\frac{1}{2N+1}.$$

As the left hand members are independent of N, we finally obtain, by increasing N indefinitely,

$$\lim_{n=\infty} U_n(x) = \frac{2}{\pi}.$$

In equation (27), we change x into $x + \pi/2$, whence

$$|\sin x| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{\nu=1}^{N}\left(\frac{1}{2\nu+1} - \frac{1}{2\nu-1}\right)\cos 2\nu x + R'_N,$$

$$|R'_N| < \frac{2}{\pi}\cdot\frac{1}{2N+1},$$

and from these formulæ we obtain, in the same way as before,

$$\lim_{n=\infty} V_n(x) = \frac{2}{\pi}.$$

§ 4. *On the range of variation of* $U_n(x)$.

On account of the obvious relations

$$U_n(x+\pi) = U_n(\pi - x) = U_n(x),$$

it is sufficient, in studying the range of variation of $U_n(x)$, to consider the interval

$$0 \leqq x \leqq \frac{2}{\pi}.$$

Let x'_n be the value of x in this interval which makes $U_n(x)$ an absolute minimum; we shall now proceed to show that

$$\lim_{n=\infty} U_n(x'_n) = \frac{1}{2}.$$

To this purpose, we begin by determining, for each positive integral n, integers $m(n)$, $l_0(n)$, $l_1(n)$, $\cdots$, $l_\lambda(n)\cdots$ such that

$$\frac{2m\pi}{2n+1} \leqq x'_n < \frac{2(m+1)\pi}{2n+1}, \tag{31}$$

and

$$l_0 = 0,$$

$$l_1 x'_n \leqq \frac{\pi}{2} < (l_1+1)x'_n,$$

$$l_2 x'_n \leqq \frac{3\pi}{2} < (l_2+1)x'_n, \tag{32}$$

. . . . . .

$$l_\lambda x'_n \leqq (2\lambda-1)\frac{\pi}{2} < (l_\lambda+1)x'_n,$$

. . . . . .

and also an index $\mu\,(n)$ such that

$$l_\mu < n \leqq l_{\mu+1}. \tag{33}$$

We then have

$$|\cos \nu x_n'| = (-1)^\lambda \cos \nu x_n' \quad \text{for} \quad l_\lambda + 1 \leqq \nu \leqq l_{\lambda+1},$$

and consequently

$$U_n(x_n') = \frac{1}{n+1}\sum_{\nu=1}^{n}|\cos \nu x_n'| = \frac{1}{n+1}\sum_{\lambda=0}^{\mu-1}(-1)^\lambda \sum_{\nu=l_\lambda+1}^{l_{\lambda+1}} \cos \nu x_n' + \frac{(-1)^\mu}{n+1}\sum_{\nu=l_\mu+1}^{n} \cos \nu x_n'$$

$$\sum_{\lambda=0}^{\mu-1}(-1)^\lambda\left(\sin\frac{2l_{\lambda+1}+1}{2}x_n' - \sin\frac{2l_\lambda+1}{2}x_n'\right)$$

$$= \frac{+(-1)^\mu\left(\sin\dfrac{2n+1}{2}x_n' - \sin\dfrac{2l_\mu+1}{2}x_n'\right)}{2(n+1)\sin\dfrac{x_n'}{2}} \tag{34}$$

$$= \frac{-\sin\dfrac{x_n'}{2} + 2\sum\limits_{\lambda=0}^{\mu-1}(-1)^\lambda \sin\dfrac{2l_{\lambda+1}+1}{2}x_n' + (-1)^\mu \sin\dfrac{2n+1}{2}x_n'}{2(n+1)\sin\dfrac{x_n'}{2}}.$$

To find the inferior limit of the expression (34) for $n = \infty$, it will be convenient to distinguish two cases:

Case 1: From the set of integers $1, 2, 3, \cdots$ we may extract a set

$$n_1^{(1)},\ n_2^{(1)},\ n_3^{(1)},\ \cdots$$

where

$$n_{\nu+1}^{(1)} > n_\nu^{(1)}, \qquad \lim_{\nu=\infty} n_\nu^{(1)} = \infty,$$

and when n assumes these values, x_n' is such that in (31) we have

$$m < n^{\frac{1}{4}}.$$

Case 2: From the set of integers $1, 2, 3, \cdots$ we may extract a set

$$n_1^{(2)},\ n_2^{(2)},\ n_3^{(2)},\ \cdots,$$

where

$$n_{\nu+1}^{(2)} > n_\nu^{(2)}, \qquad \lim_{\nu=\infty} n_\nu^{(2)} = \infty,$$

and when n assumes these values, x_n' is such that in (31) we have

$$m \eqq n^{\frac{1}{4}}.$$

Case 1: n assumes the value in the set $n_1^{(1)}, n_2^{(1)}, n_3^{(1)}, \cdots$, or briefly, the set $\{1\}$, and for all these values of n we have

$$m < n^{\frac{1}{2}}. \tag{35}$$

We shall first develop some consequences of (31), (32), (33) and (35). By the last inequality (32), we have

$$\frac{2\lambda-1}{l_\lambda+1}\frac{\pi}{2} < x_n' \lessgtr \frac{2\lambda-1}{l_\lambda}\frac{\pi}{2}, \tag{36}$$

which may also be written

$$-\frac{2\lambda-1}{l_\lambda+1}\frac{\pi}{4} < \frac{2l_\lambda+1}{2}x_n' - (2\lambda-1)\frac{\pi}{2} \lessgtr \frac{2\lambda-1}{l_\lambda}\frac{\pi}{4}. \tag{37}$$

From (31) and (36) we find

$$\frac{2\lambda-1}{l_\lambda+1} < \frac{4(m+1)}{2n+1}; \qquad \frac{2\lambda-1}{l_\lambda} \lessgtr \frac{4m}{2n+1}, \tag{38}$$

and from (37) and (38)

$$\left|\frac{2l_\lambda+1}{2}x_n' - (2\lambda-1)\frac{\pi}{2}\right| \lessgtr \frac{2\lambda-1}{l_\lambda}\frac{\pi}{4} < \frac{\pi(m+1)}{2n+1}\frac{l_\lambda+1}{l_\lambda} \lessgtr \frac{2\pi(m+1)}{2n+1}. \tag{39}$$

From the first inequality (38) we obtain, making $\lambda = \mu$ and using (33),

$$2\mu - 1 < \frac{4(l_\mu+1)(m+1)}{2n+1} \lessgtr \frac{4n}{2n+1}(m+1) < 2(m+1),$$

$$2\mu - 1 \lessgtr 2m+1,$$

$$\mu \lessgtr m+1,$$

and from the second inequality (38), making $\lambda = \mu + 1$ and using (33),

$$2\mu + 1 \gtrless \frac{4l_{\mu+1}m}{2n+1} \gtrless \frac{4nm}{2n+1} = 2m\left(1 - \frac{1}{2n+1}\right)$$

or on account of (35), for n sufficiently large

$$2\mu + 1 \gtrless 2m,$$

whence

$$2\mu + 1 \gtrless 2m+1,$$

$$\mu \gtrless m,$$

or finally, combining the two inequalities obtained,

$$m \lessgtr \mu \lessgtr m+1. \tag{40}$$

Furthermore, (32) and (33) give

$$(2\mu-1)\frac{\pi}{2} < (l_\mu+1)\,x_n' \lessgtr nx_n' \lessgtr \frac{n}{l_1}\frac{\pi}{2},$$

and

$$(2\mu+1)\frac{\pi}{2} \gtrless l_{\mu+1}\,x_n' \gtrless nx_n' > \frac{n}{l_1+1}\frac{\pi}{2} \gtrless \frac{n}{2l_1}\frac{\pi}{2},$$

whence

$$\frac{1}{4}\left(\frac{n}{l_1}-2\right) < \mu < \frac{1}{2}\left(\frac{n}{l_1}+1\right). \tag{41}$$

We now have

$$\sum_{\lambda=0}^{\mu-1}(-1)^\lambda \sin\frac{2l_{\lambda+1}+1}{2}x_n' - \sum_{\lambda=0}^{\mu-1}(-1)^\lambda \sin(2\lambda+1)\frac{\pi}{2}$$

$$= \sum_{\lambda=0}^{\mu-1}(-1)^\lambda\left(\frac{2l_{\lambda+1}+1}{2}x_n' - (2\lambda+1)\frac{\pi}{2}\right)\cos\left[(2\lambda+1)\frac{\pi}{2} + \theta_\lambda\left(\frac{2l_{\lambda+1}+1}{2}x_n' - (2\lambda+1)\frac{\pi}{2}\right)\right], \qquad (0<\theta_\lambda<1),$$

whence, by (39), (40) and (35),

$$\begin{aligned}
&\left|\sum_{\lambda=0}^{\mu-1}(-1)^\lambda \sin\frac{2l_{\lambda+1}+1}{2}x_n' - \mu\right| \\
&= \left|\sum_{\lambda=0}^{\mu-1}(-1)^\lambda \sin\frac{2l_{\lambda+1}+1}{2}x_n' - \sum_{\lambda=0}^{\mu-1}(-1)^\lambda \sin(2\lambda+1)\frac{\pi}{2}\right| \\
&\lessgtr \sum_{\lambda=0}^{\mu-1}\left|\frac{2l_{\lambda+1}+1}{2}x_n' - (2\lambda+1)\frac{\pi}{2}\right| \lessgtr \sum_{\lambda=0}^{\mu-1}\frac{2\pi(m+1)}{2n+1} = \frac{2\pi\mu(m+1)}{2n+1} \\
&\lessgtr \frac{2\pi(m+1)^2}{2n+1} < \frac{8\pi m^2}{2n} < \frac{4\pi}{n^{\frac{1}{3}}}.
\end{aligned} \tag{42}$$

Now first suppose that from the set $\{1\}$ we may detach a set $\{3\}$:

$$n_1^{(3)},\ n_2^{(3)},\ n_3^{(3)},\ \cdots;\ n_{\nu+1}^{(3)} > n_\nu^{(3)},\ \lim_{\nu=\infty} n_\nu^{(3)} = \infty,$$

such that, when n assumes the values of the set $\{3\}$.

$$\lim_{\{3\}}\frac{n}{l_1} = \infty. \tag{43}$$

We have, by (34) and (42),

$$U_n(x_n') > \frac{2\left(\mu - \frac{4\pi}{n^{\frac{1}{3}}}\right) + (-1)^\mu \sin\frac{2n+1}{2}x_n'}{2(n+1)\sin\frac{x_n'}{2}} - \frac{1}{2(n+1)}$$

$$\geqq \frac{2\left(\mu - \frac{4\pi}{n^{\frac{1}{3}}}\right) - 1}{2(n+1)\frac{x_n'}{2}} - \frac{1}{2(n+1)},$$

and by (40) and (31),

$$U_n(x_n') > \frac{2\left(m - \frac{4\pi}{n^{\frac{1}{3}}}\right) - 1}{\frac{2(n+1)(m+1)\pi}{2n+1}} - \frac{1}{2(n+1)}.$$

On account of (40) and (41), it follows from (43) that

$$\lim_{\{3\}} m = \infty,$$

and consequently the preceding inequality gives

$$\liminf_{\{3\}} U_n(x_n') \geqq \frac{2}{\pi} > \frac{1}{2}. \tag{44}$$

In the second place, suppose that from the set $\{1\}$ we may detach a set $\{4\}$

$$n_1^{(4)}, n_2^{(4)}, n_3^{(4)}, \cdots; n_{\nu+1}^{(4)} > n_\nu^{(4)}, \lim_{\nu=\infty} n_\nu^{(4)} = \infty,$$

such that, when n assumes the values of the set $\{4\}$,

$$\lim_{\{4\}} \frac{n}{l_1} = \alpha, \tag{45}$$

where α is finite. The inequalities (41) then show that for the set $\{4\}$, μ is enclosed between finite limits, and consequently we may detach from $\{4\}$ a set $\{5\}$

$$n_1^{(5)}, n_2^{(5)}, n_3^{(5)}, \cdots; n_{\nu+1}^{(5)} > n_\nu^{(5)}, \lim_{\nu=\infty} n_\nu^{(5)} = \infty,$$

such that

$$\lim_{\{5\}} \frac{n}{l_1} = \lim_{\{4\}} \frac{n}{l_1} = \alpha, \qquad \lim_{\{5\}} \mu = \mu_0, \tag{46}$$

where μ_0 is finite. The inequalities from which we derived (41) were

$$(2\mu - 1)\frac{\pi}{2} \leqq \frac{n}{l_1}\frac{\pi}{2}, \qquad (2\mu + 1)\frac{\pi}{2} > \frac{n}{l_1 + 1}\frac{\pi}{2};$$

as (41) gives

$$l_1 > \frac{n}{4\mu + 2},$$

we have for n sufficiently large in the set $\{5\}$

$$l_1 > \frac{n}{8\mu_0 + 4}, \tag{47}$$

and consequently the preceding inequalities give, when combined with (46),

$$2\mu_0 - 1 \overset{=}{<} \alpha \overset{=}{<} 2\mu_0 + 1. \tag{48}$$

From (32) we obtain

$$(2n+1)\left[\frac{\pi}{4(l_1+1)} - \frac{\pi^3}{48.8l^3}\right] < (2n+1)\left[\frac{x_n'}{2} - \frac{x_n'^3}{48}\right] < (2n+1)\sin\frac{x_n'}{2}$$
$$< (2n+1)\frac{x_n'}{2} < (2n+1)\frac{\pi}{4l_1},$$

whence by (47)

$$\lim_{\{5\}} (2n+1)\sin\frac{x_n'}{2} = \frac{\alpha\pi}{2},$$

and in the same way we find

$$\lim_{\{5\}} \sin(2n+1)\frac{x_n'}{2} = \sin\frac{\alpha\pi}{2},$$

so that, by (34), (42) and (46)

$$\lim_{\{5\}} U_n(x_n') = \frac{2\mu_0 + (-1)^{\mu_0}\sin\alpha\pi/2}{\alpha\pi/2},$$

or, making

$$\frac{\alpha\pi}{2} = \mu_0\pi + z,$$

where, according to (48), $-\pi/2 \overset{=}{<} z \overset{=}{<} \pi/2$,

$$\lim_{\{5\}} U_n(x_n') = \frac{2\mu_0 + \sin z}{\mu_0\pi + z}.$$

For $z = \pm\pi/2$, the right hand member is equal to $2/\pi$; it therefore becomes a minimum for $z = z_0$, where $-(\pi/2) < z_0 < (\pi/2)$, z_0, being a root of the equation

$$\varphi(z) = (\mu_0\pi + z)\cos z - (2\mu_0 + \sin z) = 0,$$

and the minimum is equal to

$$\frac{2\mu_0 + \sin z_0}{\mu_0\pi + z_0} = \cos z_0.$$

For $\mu_0 = 0$, we have

$$\cos z_0 = \frac{\sin z_0}{z_0} \overset{=}{>} \frac{\sin(\pi/2)}{(\pi/2)} = \frac{2}{\pi},$$

and for $\mu_0 \overset{=}{>} 1$, it is seen at once that

$$\varphi(-\pi/2) < 0, \qquad \varphi(0) > 0, \qquad \varphi(\pi/2) < 0$$

and

$$\frac{d\varphi}{dz} = -(\mu_0\pi + z)\sin z \begin{matrix} > 0 \text{ for } -\pi/2 < z < \\ < 0 \text{ for } 0 < z < \pi/2, \end{matrix}$$

so that $\varphi(z) = 0$ has one and only one root in each of the intervals $-\pi/2 < z < 0$ and $0 < z < \pi/2$. We further have, for $\mu_0 \gtrless 1$,

$$\varphi\left(-\frac{\pi}{3}\right) = \left(\mu_0 \pi - \frac{\pi}{3}\right) \cdot \frac{1}{2} - 2\mu_0 + \frac{\sqrt{3}}{2} < 0,$$

$$\varphi\left(\frac{\pi}{3}\right) = \left(\mu_0 \pi + \frac{\pi}{3}\right) \cdot \frac{1}{2} - 2\mu_0 - \frac{\sqrt{3}}{2} < 0,$$

so that $|z_0| < \pi/3$ and $\cos z_0 > \frac{1}{2}$. Consequently, we have, whatever may be the values of α and μ_0,

$$\lim_{\{s\}} U_n(x_n') > \frac{1}{2},$$

and combining this result with (44), we obtain

$$\liminf_{\{1\}} U_n(x_n') > \frac{1}{2}. \tag{49}$$

Case 2: n assumes the values in the set $\{2\}$, and for all these values of n we have

$$m \gtrless n^{\frac{1}{3}}. \tag{50}$$

Then, as $|\cos \nu x| \gtrless \cos^2 \nu x$, we have

$$U_n(x_n') \gtrless \frac{\cos^2 x + \cos^2 2x + \cdots + \cos^2 nx}{n+1} = \frac{n}{n+1}(1 - y^2),$$

where y is given by (17), and from (25) we obtain

$$U_n(x_n') \gtrless \frac{n}{n+1}\left(\frac{1}{2} - \frac{1}{4n} - \frac{2n+1}{4n} \cdot \frac{1}{4m+3}\right),$$

whence, by (50),

$$\liminf_{\{2\}} U_n(x_n') \gtrless \frac{1}{2}. \tag{51}$$

Now (11) gives

$$\lim_{n=\infty} U_n\left(\frac{\pi}{2}\right) = \frac{1}{2},$$

so that, by (49) and (51), we have

$$\lim_{n=\infty} U_n(x_n') = \frac{1}{2}.$$

On the other hand, $U_n(x)$ obviously becomes a maximum $= n/(n+1)$ for $x = 0$; thus the limiting values, for $n = \infty$, of the maxima and minima of $U_n(x)$ are included in the same range (from $\frac{1}{2}$ to 1) as are the limit values

of $U_n(x)$ for x independent of n, or in other words, $U_n(x)$ presents no analogon to Gibbs' phenomenon.

§ 5. *On the range of variation of $V_n(x)$ and an analogon to Gibbs' phenomenon.*

As we obviously have

$$V_n(x+\pi) = V_n(\pi - x) = V_n(x),$$

it is sufficient to consider the interval

$$0 \leqq x \leqq \frac{\pi}{2}.$$

We shall first prove that in the interval $0 \leqq x < \pi/n$, $V_n(x)$ has a certain maximum, which increases monotonously with n towards the limit $\sin z_0$ for $n = \infty$, where z_0 is the smallest positive root of the equation

$$z_0 = \tan \tfrac{1}{2} z_0.$$

Furthermore it will be shown that the maximum in question is the absolute maximum of $V_n(x)$.

First supposing $0 \leqq x < \pi/n$, we have

$$\begin{aligned} V_n(x) &= \frac{|\sin x| + |\sin 2x| + \cdots + |\sin nx|}{n+1} \\ &= \frac{\sin x + \sin 2x + \cdots + \sin nx}{n+1} \\ &= \frac{1}{2(n+1)}\left(\cot\frac{x}{2} - \frac{\cos\dfrac{2n+1}{2}x}{\sin\dfrac{x}{2}}\right), \end{aligned} \tag{52}$$

whence

$$\begin{aligned} \frac{dV_n(x)}{dx} &= \frac{-1 + (2n+1)\sin\dfrac{2n+1}{2}x\sin\dfrac{x}{2} + \cos\dfrac{2n+1}{2}x\cos\dfrac{x}{2}}{4(n+1)\sin^2\dfrac{x}{2}} \\ &= \frac{-1 + (n+1)\cos nx - n\cos(n+1)x}{4(n+1)\sin^2\dfrac{x}{2}} \\ &= \frac{n\sin^2\dfrac{n+1}{2}x - (n+1)\sin^2\dfrac{n}{2}x}{2(n+1)\sin^2\dfrac{x}{2}}. \end{aligned}$$

If x_n'' is the value of x, in the interval considered, which makes $V_n(x)$ a maximum, we therefore have

$$\frac{\sin\frac{n+1}{2}x_n''}{\sin\frac{n}{2}x_n''}=\sqrt{\frac{n+1}{n}},$$

where the square root is positive, as

$$\frac{n}{2}x_n''<\frac{n+1}{2}x_n''\leqq\frac{n+1}{2n}\pi\leqq\pi$$

for $n\gtreqless 1$. From the above equation we obtain

$$\frac{\sin\frac{n+1}{2}x_n''+\sin\frac{n}{2}x_n''}{\sin\frac{n+1}{2}x_n''-\sin\frac{n}{2}x_n''}=\frac{\sqrt{1+\frac{1}{n}}+1}{\sqrt{1+\frac{1}{n}}-1},$$

or

$$\tan\frac{2n+1}{4}x_n''=\frac{\sqrt{1+\frac{1}{n}}+1}{\sqrt{1+\frac{1}{n}}-1}\tan\frac{x_n''}{4}.$$

The right hand member being positive, it follows that

$$\frac{2n+1}{4}x_n''<\frac{\pi}{2}.$$

Introducing the notations

$$(53)\qquad \frac{2n+1}{2}x_n''=z<\pi,\qquad \sqrt{1+\frac{1}{n}}-1=h,\qquad \omega=\frac{h(2+h)}{2(h^2+2h+2)},$$

whence

$$\frac{x_n''}{4}=\omega z<\frac{\pi}{4},\qquad n=\frac{1}{h(2+h)},$$

the above equation becomes

$$(54)\qquad \tan\frac{z}{2}=\frac{h+2}{h}\tan\omega z\qquad (z<\pi),$$

and from (52) we obtain

$$V_n(x_n'') = \frac{1}{2(n+1)}\left(\frac{1-\tan^2\frac{x_n''}{4}}{2\tan\frac{x_n''}{4}} - \frac{1+\tan^2\frac{x_n''}{4}}{2\tan\frac{x_n''}{4}}\cdot\frac{1-\tan^2\frac{2n+1}{4}x_n''}{1+\tan^2\frac{2n+1}{4}x_n''}\right)$$

$$= \frac{1}{\frac{2}{h(2+h)}+2}\left(\frac{1-\tan^2\omega z}{2\tan\omega z} - \frac{1+\tan^2\omega z}{2\tan\omega z}\cdot\frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}}\right)$$

$$= \frac{1}{\frac{2}{h(2+h)}+2}\left(\frac{1-\frac{h^2}{(2+h)^2}\tan^2\frac{z}{2}}{\frac{2h}{2+h}\tan\frac{z}{2}} - \frac{1+\frac{h^2}{(2+h)^2}\tan^2\frac{z}{2}}{\frac{2h}{2+h}\tan\frac{z}{2}}\cdot\frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}}\right)$$

$$= \frac{1}{\frac{2}{h(2+h)}+2}\cdot\frac{1-\frac{h^2}{(2+h)^2}}{\frac{2h}{2+h}}\cdot\frac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}},$$

or finally

$$V_n(x_n'') = \frac{\sin z}{1+h}. \tag{55}$$

In order to prove that this expression increases monotonously with n, we differentiate and obtain

$$\frac{dV_n(x_n'')}{dh} = \frac{\cos z}{1+h}\frac{dz}{dh} - \frac{\sin z}{(1+h)^2},$$

or

$$\frac{dV_n(x_n'')}{dh} = \frac{2}{(1+h)\left(1+\tan^2\frac{z}{2}\right)}\left[\frac{1}{2}\left(1-\tan^2\frac{z}{2}\right)\frac{dz}{dh} - \frac{\tan\frac{z}{2}}{1+h}\right]. \tag{56}$$

Now (54) gives by differentiation

$$\left[\frac{1}{2}+\frac{1}{2}\tan^2\frac{z}{2} - \frac{2+h}{h}\omega(1+\tan^2\omega z)\right]\frac{dz}{dh}$$

$$= -\frac{2}{h^2}\tan\omega z + \frac{2+h}{h}(1+\tan^2\omega z)\,z\frac{d\omega}{dh},$$

or on substituting the value of $\tan \omega z$ given by (54) in the first member,

$$\left[\frac{1}{2}+\frac{1}{2}\tan^2\frac{z}{2}-\frac{2+h}{h}\omega-\frac{h}{2+h}\omega\tan^2\frac{z}{2}\right]\frac{dz}{dh}$$
$$=-\frac{2}{h^2}\tan\omega z+\frac{2+h}{h}(1+\tan^2\omega z)\,z\frac{d\omega}{dh},$$

or finally, introducing the value of ω given by (53) in the left hand member and in the expression $d\omega/dh$, and then making the necessary algebraic reductions,

$$\frac{1+h}{h^2+2h+2}\left(\tan^2\frac{z}{2}-1\right)\frac{dz}{dh}=-\frac{2}{h^2}\tan\omega z+\frac{2(1+h)(2+h)}{h(h^2+2h+2)^2}z(1+\tan^2\omega z),$$

whence

$$\frac{1}{2}\left(1-\tan^2\frac{z}{2}\right)\frac{dz}{dh}-\frac{\tan\frac{z}{2}}{1+h}=\left(\frac{h^2+2h+2}{h^2(1+h)}-\frac{2+h}{h(1+h)}\right)\tan\omega z-\frac{2\omega}{h^2}z(1+\tan^2\omega z)$$
$$=\frac{2}{h^2}\left[\frac{\tan\omega z}{1+h}-\omega z(1+\tan^2\omega z)\right]$$
$$=\frac{2\cos\omega z\sin\omega z-2(1+h)\omega z}{h^2(1+h)\cos^2\omega z}$$
$$=\frac{\sin 2\omega z-2\omega z(1+h)}{h^2(1+h)\cos^2\omega z}<0.$$

By comparison with (56) it then follows that

$$\frac{dV_n(x_n'')}{dh}<0,$$

so that the maximum of $V_n(x)$ for $0 \leqq nx < \pi$ increases monotonously with h decreasing, that is, with n increasing. We now have to find the limit of this maximum for $n=\infty$. To this purpose, let z_0 be one of the limit values toward which the root $z<\pi$ of (54) tends for h going towards zero by the values obtained by making $n=1, 2, 3, \cdots$ in (53). Extracting from this set of values an infinitely decreasing set such that for h passing through the latter values, z tends to the limit z_0, we have

$$\lim\tan\frac{z}{2}=\tan\frac{z_0}{2},\qquad \lim\frac{h+2}{h}\tan\omega z=z_0,$$

so that z_0 ($\leqq \pi$) satisfies the equation

$$\tan\frac{z_0}{2}=z_0. \tag{57}$$

This equation obviously has one and only one root in the interval $0 < z_0 \overline{\leqslant} \pi$, whence it follows that the root $z < \pi$ of (54) tends towards one definite limit for $n = \infty$, viz. z_0. From (55) it then follows that

$$\lim_{n=\infty} V_n(x_n'') = \sin z_0. \tag{58}$$

The numerical values are approximately

$$\begin{aligned} z_0 &= 2.33133 \cdots = 133^\circ\, 34' \cdots, \\ \sin z_0 &= 0.72457 \cdots > \frac{2}{\pi} = 0.6366 \cdots. \end{aligned} \tag{59}$$

We now proceed to the investigation of $V_n(x)$ in the interval $\pi/n \overline{\leqslant} x \overline{\leqslant} \pi/2$. Assuming that

$$m\pi \overline{\leqslant} nx < (m+1)\pi, \qquad m \overline{\geqslant} 1, \tag{60}$$

we determine integers $l_1, l_2, \cdots, l_m$ such that

$$\begin{aligned} l_1 x &\overline{\leqslant} \pi < (l_1+1)x, \\ l_2 x &\overline{\leqslant} 2\pi < (l_2+1)x, \\ &\cdots\cdots\cdots \\ l_m x &\overline{\leqslant} m\pi < (l_m+1)x, \end{aligned}$$

and obtain

$$\begin{aligned} V_n(x) &= \frac{1}{n+1}\left(\sum_{\nu=1}^{l_1} \sin x - \sum_{\nu=l_1+1}^{l_2} \sin x + \cdots + (-1)^m \sum_{\nu=l_m+1}^{n} \sin x\right) \\ &= \frac{\cos\frac{x}{2} - 2\cos\frac{2l_1+1}{2}x + 2\cos\frac{2l_2+1}{2}x - \cdots}{2(n+1)\sin\frac{x}{2}} \\ &\qquad \frac{+ (-1)^m \cdot \cos\frac{2l_m+1}{2}x + (-1)^{m+1}\cos\frac{2n+1}{2}x}{2(n+1)\sin\frac{x}{2}}, \end{aligned}$$

whence

$$V_n(x) \overline{\leqslant} \frac{2m+1+(-1)^{m+1}\cos\frac{2n+1}{2}x}{2(n+1)\sin\frac{x}{2}}. \tag{61}$$

We have

$$\begin{aligned} 2(n+1)\sin\frac{x}{2} &> 2(n+1)\left(\frac{x}{2} - \frac{x^3}{48}\right) = \frac{2n+1}{2}x + \frac{x}{2}\left(1 - \frac{(n+1)x^2}{12}\right) \\ &> \frac{2n+1}{2}x + \frac{x}{2}\left(1 - \frac{(n+1)(m+1)^2\pi^2}{12n^2}\right), \end{aligned}$$

whence

$$2(n+1)\sin\frac{x}{2} > \frac{2n+1}{2}x \tag{62}$$

for

$$n \gtrless (m+1)^2, \tag{63}$$

(except for $m=1$, $n=4$, in which case it is easily shown by numerical calculation that the absolute maximum of $V_4(x)$ occurs for an x in the interval $0<x<\pi/4$). From (61) and (62) it follows that, as soon as (63) is satisfied,

$$V_n(x) < \frac{2m+1+(-1)^{m-1}\cos\dfrac{2n+1}{2}x}{\dfrac{2n+1}{2}x},$$

or making

$$\frac{2n+1}{2}x = m\pi + z, \qquad 0 < z < \pi + \frac{m+1}{2n}\pi < \frac{3\pi}{2}, \tag{64}$$

we find

$$V_n(x) < \frac{2m+1-\cos z}{m\pi+z}, \tag{65}$$

so that an upper limit for $V_n(x)$ is furnished by the maximum of the right hand expression. In order to obtain this maximum, we must have

$$\varphi(z) = (m\pi+z)\sin z - (2m+1) + \cos z = 0. \tag{66}$$

Determining z_∞ by the conditions $\sin z_\infty = 2/\pi$, $\pi/2 < z_\infty < \pi$, we find

$$\varphi'(z) = (m\pi+z)\cos z \begin{cases} > 0, & \left(0 < z < \dfrac{\pi}{2}\right), \\ < 0, & \left(\dfrac{\pi}{2} < z < \dfrac{3\pi}{2}\right), \end{cases}$$

and

$$\varphi(0) = -2m < 0,$$

$$\varphi\left(\frac{\pi}{2}\right) = (2m+1)\left(\frac{\pi}{2}-1\right) > 0,$$

$$\varphi(z_\infty) = \frac{2}{\pi}z_\infty - 1 + \cos z_\infty < 0,$$

$$\varphi(\pi) = -(2m+2) < 0,$$

$$\varphi\left(\frac{3\pi}{2}\right) = -(2m+1)\left(\frac{\pi}{2}+1\right) - 1 < 0,$$

so that, in the interval $0 < z < 3\pi/2$, the equation (66) has one root between 0 and $\pi/2$, corresponding to a minimum, and one root z_m between $\pi/2$ and z_∞, corresponding to a maximum, which on account of (66) has the value

$$\frac{2m+1-\cos z_m}{m\pi+z_m} = \sin z_m . \tag{67}$$

On the other hand, this maximum decreases monotonously with m increasing, for we have by (66)

$$\begin{aligned}\frac{d}{dm}\frac{2m+1-\cos z_m}{m\pi+z_m} &= \frac{\partial}{\partial m}\frac{2m+1-\cos z_m}{m\pi+z_m} + \frac{\partial}{\partial z_m}\frac{2m+1-\cos z_m}{m\pi+z_m}\frac{dz_m}{dm} \\ &= \frac{\partial}{\partial m}\frac{2m+1-\cos z_m}{m\pi+z_m} \\ &= \frac{2(m\pi+z_m)-\pi(2m+1-\cos z_m)}{(m\pi+z_m)^2} \\ &= \frac{2-\pi\sin z_m}{m\pi+z_m} < 0, \text{ on account of } z_m > z_\infty .\end{aligned}$$

By numerical calculation for $m=1$, we obtain $137^\circ 35' < z_1 < 137^\circ 36'$, $\sin z_1 < 0.6745\cdots$, so that, for n satisfying (63) and any $m \gtreqless 1$,

$$V_n(x) < \sin z_1 < 0.6745 . \tag{68}$$

Now turning our attention to the remaining case $n+1 \lesseqgtr (m+1)^2$, we find from (25)

$$\begin{aligned}V_n(x) &< \frac{n}{n+1}\sqrt{\frac{2n+1}{4n}\,1+\frac{1}{4m_1+3}} \\ &= \frac{1}{2}\sqrt{\left(1+\frac{1}{4m_1+3}\right)\left(2-\frac{1}{n+1}\right)\left(1-\frac{1}{n+1}\right)},\end{aligned}$$

where m_1 is determined by

$$\frac{2m_1\pi}{2n+1} \lesseqgtr x < \frac{2(m_1+1)\pi}{2n+1},$$

so that, by (60), $m_1 = m$ or $m_1 = m+1$. In either case we have

$$\begin{aligned}V_n(x) &< \frac{1}{2}\sqrt{\left(1+\frac{1}{4m+3}\right)\left(2-\frac{1}{n+1}\right)\left(1-\frac{1}{n+1}\right)} \\ &\lesseqgtr \frac{1}{2}\sqrt{\left(1+\frac{1}{4m+3}\right)\left(2-\frac{1}{(m+1)^2}\right)\left(1-\frac{1}{(m+1)^2}\right)}.\end{aligned}$$

It is readily seen that for integral positive values of m, the last expression attains its greatest value $= 0.70904\cdots$ for $m = 6$, and by comparison with (68) we finally find that for any n and any x in the interval $\pi/\overline{\leq}\, x \,\overline{\leq}\, \pi/2$,

$$V_n(x) < 0.70904\cdots. \tag{69}$$

It is easily found that $V_n(x_n'')$, which we have shown to increase monotonously with n, exceeds the above value for $n \,\overline{\geq}\, 20$, and by numerical calculation (which may be very materially abbreviated by various artifices) it is shown that also for $n < 20$, $V_n(x_n'')$ is the absolute maximum of $V_n(x)$.

It is thus finally proved for all values of n that the absolute maximum of $V_n(x)$ occurs for an x in the interval $0 < x < \pi/n$ and increases monotonously with n toward the limit

$$\lim_{n=\infty} V_n(x_n'') = \sin z_0 = 0.72457\cdots.$$

As we have $\sin z_0 > 2/\pi$, it follows that the limit values for $n = \infty$ of the maxima and minima of $V_n(x)$ cover a wider range (from 0 to $\sin z_0$) than the limit values of $V_n(x)$ for x independent of n, which all lie within the range from 0 to $2/\pi$; the analogy to Gibbs' phenomenon is obvious.

On account of (7), we further have, for $A = 0$,

$$|s_n(x)| < M + B\sin z_0 < M + \tfrac{3}{4}B,$$

which, in this particular case, gives a still closer limitation than (9).

CHICAGO, ILL.,
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THE SOUTHERLY AND EASTERLY DEVIATIONS OF FALLING BODIES FOR AN UNSYMMETRICAL GRAVITATIONAL FIELD OF FORCE*

BY

WILLIAM H. ROEVER

Introduction and Statement of Results.

The adequacy with which theory may be made to account for a given class of physical phenomena depends largely upon the nature of the assumptions which are made in the mathematical formulation of the corresponding physical problem. This is well illustrated in the problem of the southerly deviation of falling bodies. In the treatment of this problem (and that of the easterly deviation) by GAUSS, the assumption is made that the force in the statical field of force which is at rest with respect to the earth, i. e., the field of force in which the plumb-line is in equilibrium, is constant in magnitude and direction in the neighborhood of the path of the falling body. Under this assumption,† it follows from the laws of dynamics that the southerly deviation (denoted by S. D.) is given by the formula:‡

$$\text{S. D.} = \frac{1}{3}\omega^2 \sin 2\phi \cdot \frac{h^2}{g}, \tag{1}$$

in which ω is the angular velocity of the earth's rotation, h is the height through which the body falls, and g and ϕ are the acceleration due to weight and the astronomical latitude, respectively, at the place of observation. In a recent paper by the author § it is shown that, under the assumption of a distribution of revolution (i. e., that the potential function of the earth's gravitational (or weight) field of force is of the form $f(r, z)$, where r is the distance of a general point from the earth's axis of rotation, and z is that from a fixed plane perpendicular to the axis), the same theory yields for the southerly deviation

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† Other assumptions are also made. The body is assumed to be a particle, and the effects of air currents, air resistance and the actions of the moon and sun are disregarded.

‡ See GAUSS's *Werke*, vol. 5 (1867), p. 502.

§ These Transactions, vol. 12 (1911), pp. 335–353.

the formula

$$\text{(2)} \qquad \text{S. D.} = \frac{1}{6}\left[2\omega^2 \sin 2\phi + 5\left(\frac{\partial g}{\partial x}\right)_0 \right]\frac{h^2}{g_0},$$

in which $\partial g / \partial x$ is the derivative of g along the meridian measured to the north, the subscript $(_0)$ denoting the particular value at the initial point P_0 of the falling body. It will be observed that when g is constant, $\partial g / \partial x = 0$, and therefore formula (1) is a special case of formula (2).

For the potential function for which the Besselian ellipsoid is a level surface and the formula of Helmert gives the acceleration due to weight on this ellipsoid, the following table gives a few of the corresponding values of ϕ, g, and $\partial g / \partial x$.*

TABLE I.

ϕ	g	$+10^9 \frac{\partial g}{\partial x}$
40°	980.1457	8.0399
45°	980.5966	8.1568
50°	981.0475	8.0259
55°	981.4847	7.6517
60°	981.8949	7.0463

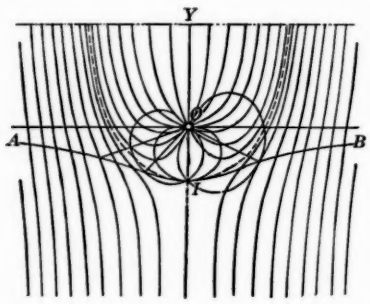
On the other hand,

$$\omega^2 = 5.3173 \cdot 10^{-9}.$$

The values given by formulas (1) and (2) are proportional to

$$2\omega^2 \sin 2\phi \qquad \text{and} \qquad 2\omega^2 \sin 2\phi + 5(\partial g / \partial x)_0.$$

Since $\sin 2\phi$ never exceeds unity, it is evident from Table I, that, for the



potential function for which the standard ellipsoid is a level surface, formula (2) gives values for the southerly deviation which are nearly five times as great as those given by formula (1).

The expression $(\partial g / \partial x)_0 / g_0$ is the curvature, at P_0, of the line of force which passes through P_0 of the statical field of force which is at rest with respect to the earth. Thus we see that the curvature of these lines of force influences

* See 2nd reference in the second following footnote.

more strongly the southerly deviation of falling bodies than does the earth's rotation.

Simple considerations show that the introduction of an attracting mass into a field of constant force, or into one whose lines of force have slight curvature, increases considerably the curvature of the lines of force in the neighborhood of the introduced mass.*

This fact suggests that the proximity, to an experiment station, of a mountain or of a mineral deposit can change considerably the curvature of the lines of force of the statical field which is at rest with respect to the earth, and thus affect considerably the southerly deviation of falling bodies. But the proximity of such a disturbing factor would make the distribution of matter which produces the earth's gravitational field of force cease to be one of revolution. Therefore it becomes necessary to derive a formula for the southerly deviation—one for the easterly deviation will also be derived—under the assumption of *a distribution not of revolution*. This has been done in this paper. (See formulas III, where $\bar{\xi}$ = S. D. and $\bar{\eta}$ = E. D.) It turns out that the first term of the expression for the southerly deviation, under this assumption, is the same as expression (2), and that the term in $h^{5/2}$, which does not appear when the assumption of a distribution of revolution is made, is negligible in comparison with the first term.

The Hungarian physicist Baron Roland Eötvös has recently devised a method by means of which he can determine experimentally the derivative $\partial g / \partial x$, and also other second derivatives of the potential function W of the force due to weight.† The following table gives the observed values of these derivatives at a few stations which are about twenty kilometers east of Arad in Hungary and just west of some mountains.

Table II.

Station number	$10^9 \frac{\partial^2 W}{\partial x \partial z}$	$10^9 \frac{\partial^2 W}{\partial y \partial z}$	$10^9 \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)$	$10^9 \frac{\partial^2 W}{\partial x \partial y}$
1,018	+32.9	+88.1	+37.0	+ 3.8
1,032	+25.8	+62.8	+72.8	+ 2.5
1,035	+15.5	+69.8	+25.5	− 4.6
2,159	−37.0	+34.5	+34.3	−13.5
2,188	+77.8	+59.8	+21.5	+30.3
2,192	+82.3	+64.5	+28.2	+26.7

* The opposite figure represents the lines of force of the gravitational field which is produced by the introduction of a mass m, at 0, into a field which originally was constant (and of which the lines of force were straight lines parallel to OY). The line AIB is the locus along which the curvature has not been changed; in this case, it is the locus of the points of inflection of the lines of force of the new field of force.

† See Encyklopädie der mathematischen Wissenschaften, Band VIIb, § 23, p. 166. Also, Verhandlungen der fünfzehnten allgemeinen Conferenz der internationalen Erdmessung, 1 (1906), pp. 337–395.

Here x is measured to the north, y to the east, and z downward along the vertical. Hence

$$g = \frac{\partial W}{\partial z}, \qquad \frac{\partial g}{\partial x} = \frac{\partial^2 W}{\partial x \partial z}, \qquad \frac{\partial g}{\partial y} = \frac{\partial^2 W}{\partial y \partial z}.$$

For the latitude (46° 10′) of this region, the normal values (i. e., the values corresponding to the potential function for which the standard ellipsoid is a level surface) of the quantities given in Table II are

$$8.1, \quad 0, \quad 4.9, \quad 0,$$

respectively.

At station 2,192 the value of the expression $2\omega^2 \sin 2\phi + 5\,(\partial g/\partial x)_0$ is about $422 \cdot 10^{-9}$, and hence formula (2) gives a value *forty times* as great as formula (1). On the other hand, at station 2,159, $2\omega^2 \sin 2\phi + 5\,(\partial g/\partial x)_0$ is $-174 \cdot 10^{-9}$, and hence formula (2) gives a *northerly deviation* which is more than sixteen times as great as the southerly deviation given by formula (1). The local value of the derivative $\partial g/\partial x = \partial^2 W/\partial x \partial z$ thus appears to determine the extent and the sign of the southerly deviation of falling bodies.

The easterly deviation of falling bodies is also influenced by the proximity of mountains or of mineral deposits. In fact, such disturbing factors yield the second term of the expression for the easterly deviation (denoted by E. D.). See formulas III.

$$\text{(3)} \qquad \text{E. D.} = \frac{2}{3}\sqrt{2}\omega \cos\phi \frac{h^{3/2}}{g_0^{1/2}} - \frac{5}{6}\frac{(\partial g/\partial y)_0}{g_0} h^2.$$

From Table II and the form of the second term in formula (3), it is evident that the effect of such disturbing factors is comparable in magnitude to the southerly deviation, but small in comparison with the first term of formula (3). The effect is also comparable in magnitude to the effect which Gauss finds is due to air resistance. Thus, for the data of Benzenberg's experiment in St. Michael's Tower, namely $\phi = 53°\ 33'$ and $h = 235$ feet, the formula of Gauss for the easterly deviation (which is equivalent to the first term of formula (3)) gives the value 3.91 lines.* When air resistance is taken into account, Gauss finds for the same data the deviation 3.86 lines. Hence the effect of air resistance is $-.05$ lines. On the other hand the value of the second term of formula (3) for the data of station 1018 (Table II) is $-.02$ lines. Gauss states that for the southerly deviation the effect of air resistance is negligible.

In this paper the effects of air resistance, air currents, and the actions of the moon and sun have not been taken into consideration.† While the com-

* 144 lines = 1 foot.

† Another matter which has not been taken into consideration in this paper is the weight of the string which supports the plumb-bob. In a field of force in which the lines of force are not rectilinear, this string will have some curvature. Therefore the position of a plumb-bob which is supported by a string with weight will be slightly different from that of a bob which is supported by a weightless string of the same length as the heavy string.

bined effects of these influences may be appreciable, it seems desirable to do first what has been done in this paper, namely to determine the effect of a local irregularity in the earth's gravitational field of force. *The surprising conclusion is reached that known local irregularities in the earth's gravitational field of force (caused by the presence of mountains or of mineral deposits or even by large buildings or tunnels) (1) influence the southerly deviation to the extent of from − 16 times to + 40 times* the amount which is given by the formula of Gauss for the southerly deviation (formula (1)), (2) affect the easterly deviation by amounts which are comparable with the effect which Gauss finds is due to air resistance.*

§ 1. *Definitions of, and formulas for, the easterly and southerly deviations of falling bodies.*

In experiments for the determination of the easterly and southerly deviations of falling bodies, a plumb-line $P_0 R$ is supported at the point P_0 from which a spherical body is later permitted to fall. On the horizontal plane which passes through the plumb-bob R, the easterly and southerly directions RE_R and RS_R are drawn (Fig. 3). The plumb-line RP_0 is the vertical at R.† The falling body, after being released from the position P_0, moves in a path which, with respect to the axes $R - E_R$, S_R, P_0, is the curve c. This curve pierces the horizontal plane $E_R RS_R$ in the point C. *The distance by which C lies to the south of RE_R is called the southerly deviation of the falling body, and that by which it lies to the east of RS_R is called the easterly deviation.*

In order to get expressions for these deviations the curve c is referred to the axes $P_0 - \bar{\eta}$, $\bar{\xi}$, $\bar{\zeta}$ which pass through P_0 and are parallel to the axes $R - E_R$, S_R, P_0.‡ The equations of the curve c are (by Eqs. (17) § 7)

$$\bar{\eta} = - W^0_\zeta \left\{ \frac{1}{3}\omega \cos\phi \cdot t^3 + \frac{5}{24} W^0_{\eta\zeta} \cdot t^4 + \frac{1}{60}\omega \left(\sin\phi W^0_{\xi\zeta} + \cos\phi \left(W^0_{\zeta\zeta} + W^0_{\eta\eta} - 4\omega^2\right)\right) t^5 \right\},$$

$$\text{I.} \quad \bar{\xi} = - W^0_\zeta \left\{ \frac{1}{24}\left(4\omega^2 \sin\phi \cos\phi + 5W^0_{\xi\zeta}\right) t^4 + \frac{1}{60}\omega \left(9 \sin\phi W^0_{\eta\zeta} + \cos\phi W^0_{\xi\eta}\right) t^5 \right\},$$

$$\bar{\zeta} = \frac{1}{2} W^0_\zeta \left\{ t^2 + \frac{1}{12}\left(W^0_{\zeta\zeta} - 4\omega^2 \cos^2\phi\right) t^4 - \frac{1}{3}\omega \cos\phi \cdot W^0_{\eta\zeta} \cdot t^5 + \cdots \right\},$$

* In the region of the Alps this range of values would probably be much greater. See note by M. BRILLOUIN on the curvature of the geoid in the Simplon tunnel, C o m p t e s r e n d u s, vol. 102 (1906), p. 916.

† It is assumed that the plumb-bob is a heavy particle and that the line is weightless and perfectly flexible.

‡ The axes $P_0 - \bar{\eta}$, $\bar{\xi}$, $\bar{\zeta}$ do not coincide with the axes $P_0 - \eta$, ξ, ζ. The latter axes are the easterly, southerly and vertical directions at P_0.

where t represents the time measured from the instant the body begins to fall, ω represents the angular velocity of the earth's rotation, ϕ the latitude of the point P_0, W the potential function of the statical field of force which is at rest with respect to the rotating earth, and the symbols W^0_ζ, $W^0_{\eta\zeta}$, $W^0_{\xi\zeta}$, $W^0_{\zeta\zeta}$, $W^0_{\eta\eta}$, $W^0_{\xi\eta}$, stand for the values of the derivatives

$$\frac{\partial W}{\partial \zeta}, \quad \frac{\partial^2 W}{\partial \eta \partial \zeta}, \quad \frac{\partial^2 W}{\partial \xi \partial \zeta}, \quad \frac{\partial^2 W}{\partial \zeta^2}, \quad \frac{\partial^2 W}{\partial^2 \eta}, \quad \frac{\partial^2 W}{\partial \xi \partial \eta}$$

at the point P_0. If we put $-\bar{\zeta} = h = RP_0$, where h is the height through which the body falls, the corresponding values of $\bar{\eta}$ and $\bar{\xi}$ are respectively the easterly and the southerly deviations which correspond to h.

Let us now express $\bar{\eta}$ and $\bar{\xi}$ explicitly in terms of h. The last equation assumes the form

$$\frac{h}{\alpha} = t^2 + \gamma t^4 + \delta t^5 + \cdots,$$

where

$$\alpha = -\frac{1}{2} W^0_\zeta, \qquad \gamma = \frac{1}{12}(W^0_{\zeta\zeta} - 4\omega^2 \cos^2\phi), \qquad \delta = -\frac{1}{3}\omega \cos\phi \cdot W^0_{\eta\zeta}.$$

Hence

$$\pm\sqrt{\frac{h}{\alpha}} = \sqrt{t^2 + \gamma t^4 + \delta t^5 + \cdots}$$

$$= t + \frac{1}{2}\gamma t^3 + \frac{1}{2}\delta t^4 + \cdots,$$

and this when solved for t yields the relation

$$t = \left(\pm\sqrt{\frac{h}{\alpha}}\right) - \frac{1}{2}\gamma\left(\pm\sqrt{\frac{h}{\alpha}}\right)^3 - \frac{1}{2}\delta\left(\pm\sqrt{\frac{h}{\alpha}}\right)^4 + \cdots.$$

Since t is positive, the upper sign must be used. The expressions for η and $\bar{\xi}$ then assume the forms:

II.
$$\bar{\eta} = \frac{2}{3}\sqrt{2} \cdot \omega \cos\phi \cdot \frac{h^{3/2}}{\sqrt{-W^0_\zeta}} + \frac{5}{6}\frac{W^0_{\eta\xi}}{(-W^0_\zeta)}h^2$$
$$+ \frac{\sqrt{2}}{30}\omega(2\sin\phi \cdot W^0_{\xi\zeta} + 2\cos\phi \cdot W^0_{\eta\eta} - 3\cos\phi \cdot W^0_{\zeta\zeta}$$
$$+ 4\omega^2(5\cos^3\phi - 2\cos^2\phi))\frac{h^{5/2}}{(-W^0_\zeta)^{3/2}},$$

$$\xi = \frac{1}{6}(4\omega^2 \sin\phi\cos\phi + 5W^0_{\xi\zeta})\frac{h^2}{(-W^0_\zeta)}$$
$$+ \frac{\sqrt{2}}{15}\omega(9\sin\phi\, W^0_{\eta\zeta} + \cos\phi \cdot W^0_{\xi\eta})\frac{h^{5/2}}{(-W^0_\zeta)^{3/2}},$$

and the expression for $\bar{\zeta}$ assumes the form $\bar{\zeta} = -h$, as it should. Since

W_η, W_ξ, W_ζ are the rectangular components of the acceleration g due to weight, we may write

$$-W^0_\zeta = g_0, \qquad W^0_{\eta\zeta} = -\left(\frac{\partial g}{\partial \eta}\right)_0, \qquad W^0_{\xi\zeta} = -\left(\frac{\partial g}{\partial \xi}\right)_0, \qquad W^0_{\zeta\zeta} = -\left(\frac{\partial g}{\partial \zeta}\right)_0,$$

and therefore the above formulas may be written as follows:

III.
$$\bar{\eta} = \frac{2}{3}\sqrt{2}\cdot\omega\cos\phi\cdot\frac{h^{3/2}}{g_0^{1/2}} - \frac{5}{6}\frac{(\partial g/\partial\eta)_0}{g_0}h^2 + \frac{\sqrt{2}}{30}\omega\left[3\cos\phi\left(\frac{\partial g}{\partial\zeta}\right)_0 - 2\sin\phi\left(\frac{\partial g}{\partial\xi}\right)_0 + 2\cos\phi\left(\frac{\partial W_\eta}{\partial\eta}\right)_0 + 4\omega^2(5\cos^3\phi - 2\cos^2\phi)\right]\frac{h^{5/2}}{g_0^{3/2}},$$

$$\bar{\xi} = \frac{4\omega^2\sin\phi\cos\phi - 5(\partial g/\partial\xi)_0}{6g_0}h^2 + \frac{\sqrt{2}}{15}\omega\left[\cos\phi\left(\frac{\partial W_\xi}{\partial\eta}\right)_0 - 9\sin\phi\left(\frac{\partial g}{\partial\eta}\right)_0\right]\frac{h^{5/2}}{g_0^{3/2}}.$$

For a distribution of revolution we have

$$\frac{\partial^2 W}{\partial\eta\,\partial\zeta} = 0, \qquad \frac{\partial^2 W}{\partial\eta\,\partial\xi} = 0,$$

and therefore the term in h^2 in the expression ($\bar{\eta}$) for the easterly deviation and the term in $h^{5/2}$ in expression ($\bar{\xi}$) for the southerly deviation drop out.

The establishment of the general equations I.

§ 2. *The curve c.* The curve c has already been defined as the path of the falling particle with respect to the rotating earth. Let us first refer it to a set of rectangular axes $O-x, y, z$, fixed in the rotating earth, and such that the origin O is a fixed point (interior to the earth) of the earth's axis a of rotation. Oz is coincident with a, and positive in the direction of the north pole, Ox is perpendicular to Oz and is so chosen that the initial point P_0 lies in the plane zOx, the positive direction being that from Oz to P_0, and finally Oy is perpendicular to Oz and Ox and is positive in the direction which Ox would have if it were revolved around the axis Oz through an angle of 90° in the direction of the earth's rotation. (See Fig. 1.)

If we represent by ω the angular velocity of the earth's rotation, and by U the function $f(x, y, z)$ which is defined by the integral:

$$U = \int_\tau \frac{dm}{\rho}, \qquad \rho = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2},$$

where dm represents that element of mass of the earth which is situated at

the point whose coordinates, with respect to the axes $O - x, y, z$, are x_i, y_i, z_i, the integration being extended throughout the whole volume of the earth, then the function:

$$W = U + \frac{\omega^2}{2}(x^2 + y^2) \tag{1}$$

is the potential function of the statical field of force which is at rest with respect to the rotating earth* (i. e., the lines of force:

$$dx : dy : dz = \frac{\partial W}{\partial x} : \frac{\partial W}{\partial y} : \frac{\partial W}{\partial z}$$

of this field are at rest with respect to the rotating earth †).

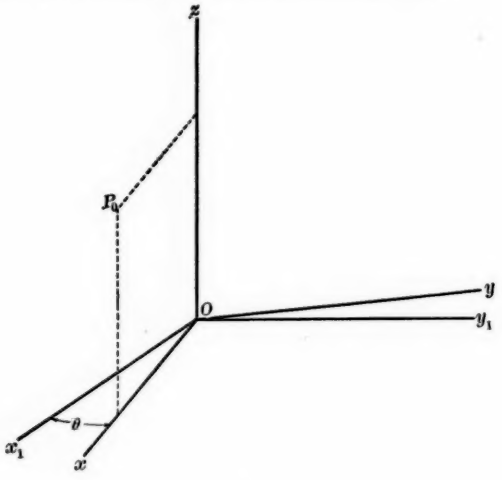


FIG. 1.

In order to find the differential equations of motion, let us denote by $O - x_1, y_1, z$ a set of rectangular axes which are fixed in space and so oriented that at the instant $t = 0$, when a falling particle leaves P_0 (at rest with respect to the earth), the set of axes $O - x, y, z$ coincides with the set $O - x_1, y_1, z$ (see Fig. 1). If we denote by θ the angle x_1Ox, positive in the direction in which the earth rotates, the relations between the coordinates are:

$$x = x_1 \cos\theta + y_1 \sin\theta, \qquad y = -x_1 \sin\theta + y_1 \cos\theta, \qquad z = z_1.$$

Then

$$U = f(x, y, z) = f_1(x_1, y_1, z, t),$$

* See PIZZETTI, *Trattato di Geodesia teoretica*, § 2 (1905).

† It should be borne in mind that the level (equipotential) surfaces, W = const., are not necessarily surfaces of revolution.

where $\theta = \omega t$ is a function of the time t. The differential equations of motion with respect to the fixed axes $O - x_1, y_1, z$, are

$$\frac{d^2 x_1}{dt^2} = \frac{\partial U}{dx_1}, \qquad \frac{d^2 y_1}{dt^2} = \frac{\partial U}{\partial y_1}, \qquad \frac{\partial^2 z}{dt^2} = \frac{\partial U}{\partial z}.$$

If we subject these differential equations to the inverse of the transformation just given, namely to the transformation

$$x_1 = x \cos\theta - y \sin\theta, \qquad y_1 = x \sin\theta + y \cos\theta, \qquad z_1 = z,$$

we obtain a set of differential equations* which is equivalent to the set:

$$\frac{d^2 x}{dt^2} - 2\omega \frac{dy}{dt} - \omega^2 x = \frac{\partial U}{\partial x}, \qquad \frac{d^2 y}{dt^2} + 2\omega \frac{dx}{dt} - \omega^2 y = \frac{\partial U}{\partial y}, \qquad \frac{d^2 z}{dt^2} = \frac{\partial U}{\partial z}.$$

Since by equation (1),

$$\frac{\partial U}{\partial x} + \omega x = \frac{\partial W}{\partial x}, \qquad \frac{\partial U}{\partial y} + \omega y = \frac{\partial W}{\partial y}, \qquad \frac{\partial U}{\partial z} = \frac{\partial W}{\partial z},$$

these equations assume the forms:

$$(2) \qquad \frac{d^2 x}{dt^2} - 2\omega \frac{dy}{dt} = \frac{\partial W}{\partial x}, \qquad \frac{d^2 y}{dt^2} + 2\omega \frac{dx}{dt} = \frac{\partial W}{\partial y}, \qquad \frac{d^2 z}{dt^2} = \frac{\partial W}{\partial z}.$$

The curve c is then that solution of the differential equations (2) which is subject to the initial conditions:

(3) When $t = 0, \quad x = r_0, \quad y = 0, \quad z = z_0, \quad \frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0.$

§ 3. *The curve c referred to the cardinal axes at P_0.* Before we can define the cardinal axes it will be necessary to define the following terms. At a general point P which is at rest with respect to the rotating earth, *the vertical* is the straight line which coincides with the plumb-line,† the plumb-bob of

* $\frac{d^2 x}{dt^2}\cos\theta - \frac{d^2 y}{dt^2}\sin\theta - 2\left(\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta\right)\omega - (x\cos\theta - y\sin\theta)\,\omega^2 = \frac{\partial U}{\partial x}\cos\theta - \frac{\partial U}{\partial y}\sin\theta,$

$\frac{d^2 x}{dt^2}\sin\theta + \frac{d^2 y}{dt^2}\cos\theta + 2\left(\frac{dx}{dt}\cos\theta - \frac{dy}{dt}\sin\theta\right)\omega - (x\sin\theta + y\cos\theta)\,\omega^2 = \frac{\partial U}{\partial x}\sin\theta + \frac{\partial U}{\partial y}\cos\theta.$

If we multiply the first of these equations by $\cos\theta$ and the second by $\sin\theta$ and add, we obtain

$$\frac{d^2 x}{dt^2} - 2\omega \frac{dy}{dt} - \omega^2 x = \frac{\partial U}{\partial x},$$

and if we multiply the first by $-\sin\theta$ and the second by $\cos\theta$ and add, we obtain

$$\frac{d^2 y}{dt^2} + 2\omega \frac{dx}{dt} - \omega^2 y = \frac{\partial U}{\partial y}.$$

† It is assumed that the plumb-bob is a heavy particle and that the line is weightless and perfectly flexible.

which is situated at P (this line does not necessarily intersect the axis of rotation a of the earth), *the meridian plane* is the plane which passes through the vertical and is parallel to the axis a, *the horizontal plane* is the plane which passes through P and is perpendicular to the vertical, the *east-and-west line* is the straight line which passes through P and is perpendicular to the meridian plane, the *north-and-south line* is the line of intersection of the meridian and the horizontal planes. By *the cardinal axes at the point P* we shall mean the vertical, the east-and-west and the north-and-south lines at the point P.

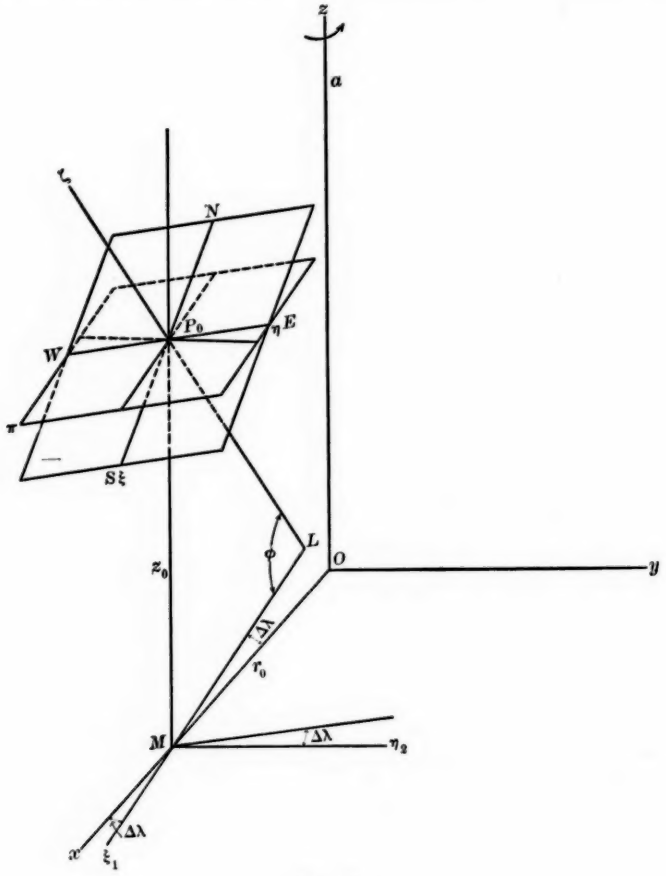


FIG. 2.

The positive directions of these axes shall be those toward the zenith, the east, and the south respectively. A few more terms will now be defined. *The*

latitude at P is the complement of the angle which the positive direction of the vertical at P makes with the positive direction of the axis a. *The longitude at P* is the angle which the meridian plane at P makes with a fixed plane through the axis a (we shall here assume that the fixed plane is the plane zOx) measured from 0° to 360° to the east.*

Let us now take the cardinal axes at the point P_0 as the axes of the coordinates ξ, η, ζ, where ξ is measured along the north-and-south line, η along the east-and-west line and ζ along the vertical. See Fig. 2, in which $Oz \equiv a$ represents the axes of rotation of the earth, and Ox, Oy a pair of

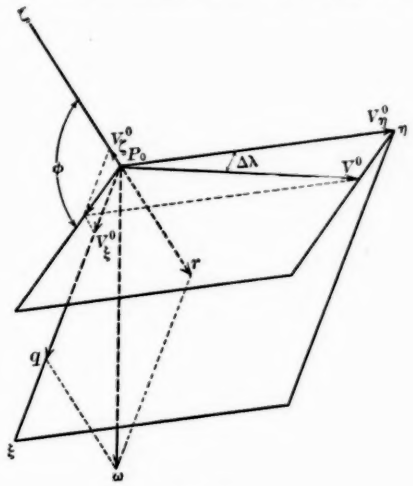


FIG. 2'.

rectangular axes which are at rest with respect to the rotating earth and are perpendicular to Oz at O. P_0 is a point of the plane xOz. $P_0 M$ is parallel to a, and intersects the plane xOy in M. We put

$$r_0 = OM, \qquad z_0 = MP_0.$$

The line $P_0 L$ is the *vertical at* P_0; it intersects the plane xOy in L. $LP_0 M$ is then the *meridian plane at* P_0. The plane $NESW$, which passes through P_0 and is perpendicular to $P_0 L$, is the *horizontal plane at* P_0. These two planes intersect in the *north-and-south-line* NS. Let us denote by π the plane which passes through P_0 and is aparallel to the plane xOy. This plane intersects the horizontal plane in the *east-and-west line* EW.

We agreed to take the plane xOz as the plane from which to measure longitude. Therefore the *longitude of* P_0 *is the angle* $\Delta\lambda = \measuredangle OML$. *The latitude of* P_0 *is the angle* $\phi = \measuredangle MLP_0$.

* See PIZZETTI, loc. cit., § 5.

In order to get the equations of transformation from the axes $O - x, y, z$ to the axes $P_0 - \xi, \eta, \zeta$, let us introduce the auxiliary variables $\xi_1, \zeta_1, \xi_2, \eta_2$, where ξ_1 represents distance measured along ML from M in the direction LM, and ζ_1, ξ_2 and η_2 are defined by the equations

$$x = \xi_2 + r_0, \qquad y = \eta_2, \qquad z = \zeta_1 + z_0.$$

Then

$$\xi_2 = \xi_1 \cos \Delta\lambda - \eta \sin \Delta\lambda, \qquad \eta_2 = \xi_1 \sin \Delta\lambda + \eta \cos \Delta\lambda,$$

and

$$\xi_1 = \zeta \cos \phi + \xi \sin \phi, \qquad \zeta_1 = \zeta \sin \phi - \xi \cos \phi.$$

Therefore

$$\begin{aligned} x &= r_0 + (\zeta \cos \phi + \xi \sin \phi) \cos \Delta\lambda - \eta \sin \Delta\lambda, \\ y &= \quad (\zeta \cos \phi + \xi \sin \phi) \sin \Delta\lambda + \eta \cos \Delta\lambda, \\ z &= z_0 + \zeta \sin \phi - \xi \cos \phi. \end{aligned} \tag{4}$$

The inverse of this transformation is the transformation:

$$\begin{aligned} \xi &= [(x - r_0) \cos \Delta\lambda + y \sin \Delta\lambda] \sin \phi - (z - z_0) \cos \phi, \\ \eta &= -(x - r_0) \sin \Delta\lambda + y \cos \Delta\lambda, \\ \zeta &= [(x - r_0) \cos \Delta\lambda + y \sin \Delta\lambda] \cos \phi + (z - z_0) \sin \phi. \end{aligned}$$

If now we subject the system of differential equations (2) to the transformation (4), we obtain the system of differential equations,*

* The details of this process are the following:

$$\frac{d^2x}{dt^2} - 2\omega \frac{dy}{dt} = \frac{\partial W}{\partial x}$$

becomes

$$\cos \Delta\lambda \left(\frac{d^2\zeta}{dt^2} \cos \phi + \frac{d^2\xi}{dt^2} \sin \phi \right) - \sin \Delta\lambda \frac{d^2\eta}{dt^2} - 2 \sin \Delta\lambda \left(\frac{d\zeta}{dt} \cos \phi + \frac{d\xi}{dt} \sin \phi \right) \omega - 2 \cos \Delta\lambda \cdot \frac{d\eta}{dt} \cdot \omega$$
$$= \frac{\partial W}{\partial \xi} \sin \phi \cos \Delta\lambda - \frac{\partial W}{\partial \eta} \sin \Delta\lambda + \frac{\partial W}{\partial \zeta} \cos \phi \cos \Delta\lambda$$

$$\frac{d^2y}{dt^2} + 2\omega \frac{dx}{dt} = \frac{\partial W}{\partial y}$$

becomes

$$\sin \Delta\lambda \left(\frac{d^2\zeta}{dt^2} \cos \phi + \frac{d^2\xi}{dt^2} \sin \phi \right) + \cos \Delta\lambda \frac{d^2\eta}{dt^2} + 2 \cos \Delta\lambda \left(\frac{d\zeta}{dt} \cos \phi + \frac{d\xi}{dt} \sin \phi \right) \omega - 2 \sin \Delta\lambda \frac{d\eta}{dt} \omega$$
$$= \frac{\partial W}{\partial \xi} \sin \phi \sin \Delta\lambda + \frac{\partial W}{\partial \eta} \cos \Delta\lambda + \frac{\partial W}{\partial \zeta} \cos \phi \sin \Delta\lambda.$$

$$\frac{d^2z}{dt^2} = \frac{\partial W}{\partial z}$$

becomes

$$\frac{\partial^2\zeta}{dt^2} \sin \phi - \frac{d^2\xi}{dt^2} \cos \phi = -\frac{\partial W}{\partial \xi} \cos \phi + \frac{\partial W}{\partial \zeta} \sin \phi.$$

If we multiply these equations respectively by $\cos \Delta\lambda \sin \phi$, $\sin \Delta\lambda \sin \phi$, $-\cos \phi$ and add, we obtain the first of the equations (5); if we multiply respectively by $\cos \Delta\lambda \cos \phi$, $\sin \Delta\lambda \cos \phi$, $\sin \phi$, and add, we obtain the second of the equations (5); and if we multiply respectively by $-\sin \Delta\lambda$, $\cos \Delta\lambda$, 0, we obtain the last of the equations (5).

$$\frac{d^2\xi}{dt^2} - 2\omega \sin\phi \cdot \frac{d\eta}{dt} = \frac{\partial W}{\partial \xi},$$

$$\frac{d^2\eta}{dt^2} + 2\omega\left(\sin\phi \cdot \frac{d\xi}{dt} + \cos\phi \cdot \frac{d\zeta}{dt}\right) = \frac{\partial W}{\partial \eta}, \tag{5}$$

$$\frac{d^2\zeta}{dt^2} - 2\omega \cos\phi \cdot \frac{d\eta}{dt} = \frac{\partial W}{\partial \zeta}.$$

The initial conditions (3) assume the form:

(6) When $t = 0, \quad \xi = \eta = \zeta = 0, \quad \frac{d\xi}{dt} = \frac{d\eta}{dt} = \frac{d\zeta}{dt} = 0.$

Therefore the curve c is that solution of the differential equations (5) which is subject to the initial conditions (6).

§ 4. *Another derivation of equations* (5). The differential equations (5) may also be obtained as follows. The projections on the moving axes η, ξ, ζ of the absolute velocity V_a are *

$$V_{a\eta} = \frac{d\eta}{dt} + V^0_\eta + q\zeta - r\xi,$$

$$V_{a\xi} = \frac{d\xi}{dt} + V^0_\xi + r\eta - p\zeta,$$

$$V_{a\zeta} = \frac{d\zeta}{dt} + V^0_\zeta + p\xi - q\eta,$$

where V^0_η, V^0_ξ, V^0_ζ are the projections on the axes η, ξ, ζ of the absolute velocity V^0 of the initial point P_0, and p, q, r are the projections on the same axes of the rotation ω of the earth. From Fig. 2′ it is easily seen that

$$V^0_\eta = V^0 \cos\Delta\lambda, \quad V^0_\xi = V^0 \sin\Delta\lambda \sin\phi, \quad V^0_\zeta = V^0 \sin\Delta\lambda \cos\phi,$$

where

$$V^0 = \omega r_0,$$

and

$$p = 0, \qquad q = \omega\cos\phi, \qquad r = -\omega\sin\phi.$$

Therefore

$$V_{a\eta} = \frac{d\eta}{dt} + \omega r_0 \cos\Delta\lambda + \omega\cos\phi \cdot \zeta + \omega\sin\phi \cdot \xi,$$

$$V_{a\xi} = \frac{d\xi}{dt} + \omega r_0 \sin\Delta\lambda \sin\phi - \omega\sin\phi \cdot \eta, \tag{7}$$

$$V_{a\zeta} = \frac{d\zeta}{dt} + \omega r_0 \sin\Delta\lambda \cos\phi - \omega\cos\phi \cdot \eta.$$

* See APPELL, *Traité de Mécanique Rationnelle*, vol. 1, (1902) §61.

The projections on the moving axes η, ξ, ζ of the absolute acceleration J_a are

$$J_{a\eta} = \frac{dV_{a\eta}}{dt} + qV_{a\zeta} - rV_{a\xi},$$

$$J_{a\xi} = \frac{dV_{a\xi}}{dt} + rV_{a\eta} - pV_{a\zeta},$$

$$J_{a\zeta} = \frac{dV_{a\zeta}}{dt} + pV_{a\xi} - qV_{a\eta}.$$

Hence by the expressions for $V_{a\eta}$, $V_{a\xi}$, $V_{a\zeta}$,

$$J_{a\eta} = \frac{d^2\eta}{dt^2} + 2\omega\left(\cos\phi \cdot \frac{d\zeta}{dt} + \sin\phi\frac{d\xi}{dt}\right) - \omega^2(\eta - r_0\sin\Delta\lambda),$$

$$(8)\quad J_{a\xi} = \frac{d^2\xi}{dt^2} - 2\omega\sin\phi \cdot \frac{d\eta}{dt} - \omega^2\sin\phi(\cos\phi\cdot\zeta + \sin\phi\cdot\xi + r_0\cos\Delta\lambda),$$

$$J_{a\zeta} = \frac{d^2\zeta}{dt^2} - 2\omega\cos\phi \cdot \frac{d\eta}{dt} - \omega^2\cos\phi(\cos\phi\cdot\zeta + \sin\phi\cdot\xi + r_0\cos\Delta\lambda).$$

Since gravitation acts instantaneously, the projections of the absolute force on the axes η, ξ, ζ are

$$\frac{\partial U}{\partial\eta}, \qquad \frac{\partial U}{\partial\xi}, \qquad \frac{\partial U}{\partial\zeta}.$$

Hence the differential equations of motion are

$$(9)\qquad J_{a\eta} = \frac{\partial U}{\partial\eta}, \qquad J_{a\xi} = \frac{\partial U}{\partial\xi}, \qquad J_{a\zeta} = \frac{\partial U}{\partial\zeta},$$

where the expressions for $J_{a\eta}$, $J_{a\xi}$, $J_{a\zeta}$ are given by (8). In virtue of relations (1) and (4) we find that

$$\frac{\partial W}{\partial\eta} = \frac{\partial U}{\partial\eta} + \omega^2(\eta - r_0\sin\Delta\lambda),$$

$$\frac{\partial W}{\partial\xi} = \frac{\partial U}{\partial\xi} + \omega^2\sin\phi(\cos\phi\cdot\zeta + \sin\phi\cdot\xi + r_0\cos\Delta\lambda),$$

$$\frac{\partial W}{d\zeta} = \frac{\partial U}{\partial\zeta} + \omega^2\cos\phi(\cos\phi\cdot\zeta + \sin\phi\cdot\xi + r_0\cos\Delta\lambda),$$

and hence equations (9) are identical with equations (5).

§ 5. *Integration of the differential equations of motion* (5). Let us write equations (5) in the following form:

$$(\alpha)\qquad \begin{aligned} &\eta'' + 2\omega(\sin\phi\cdot\xi' + \cos\phi\cdot\zeta') - W_\eta = 0,\\ &\xi'' - 2\omega\sin\phi\cdot\eta' - W_\xi = 0,\\ &\zeta'' - 2\omega\cos\phi\cdot\eta' - W_\zeta = 0, \end{aligned}$$

where the primes (′) and seconds (″) denote the first and second derivatives, with respect to the time t, of the functions to which they are attached, and the subscripts $_{(\eta, \xi, \zeta)}$ denote the first partial derivatives of W with respect to the attached subscript. In the equations which follow thirds (‴), etc., and double subscript $_{(\eta\eta, \eta\xi \cdots)}$, etc., will be used to denote the higher derivatives. For the particular values of η, ξ, ζ, and their derivatives with respect to t, which correspond to $t = 0$, we shall use the symbols η_0, ξ_0, ζ_0, η_0', ξ_0', ζ_0', η_0'', ξ_0'', etc. Hence condition (6) may be written in the following form:

$$(\beta) \qquad \eta_0 = \xi_0 = \zeta_0 = 0, \qquad \eta_0' = \xi_0' = \zeta_0' = 0.$$

For the particular values of the derivatives of W which correspond to $\eta = \eta_0$, $\xi = \xi_0$, $\zeta = \zeta_0$, we shall use the symbols W_η^0, W_ξ^0, W_ζ^0, $W_{\eta\eta}^0$, $W_{\eta\xi}^0$, $W_{\eta\zeta}^0$, $W_{\xi\xi}^0$, $W_{\xi\zeta}^0$, $W_{\zeta\zeta}^0$, etc.

For a set of functions

$$\eta = \lambda(t), \qquad \xi = \mu(t), \qquad \zeta = \nu(t),$$

which is a solution of the set of differential equations (α), the left hand member of each of the equations (α) vanishes identically. Therefore, for this solution, the derivative with respect to t of each of these left-hand members must vanish identically. Hence the identities:

$$(\alpha') \qquad \begin{aligned} &\eta''' + 2\omega(\sin\phi \cdot \xi'' + \cos\phi \cdot \zeta'') - [W_{\eta\eta} \cdot \eta' + W_{\eta\xi} \cdot \xi' + W_{\eta\zeta} \cdot \zeta'] = 0, \\ &\xi''' - 2\omega \sin\phi \cdot \eta'' \qquad\qquad - [W_{\xi\eta} \cdot \eta' + W_{\xi\xi} \cdot \xi' + W_{\xi\zeta} \cdot \zeta'] = 0, \\ &\zeta''' - 2\omega \cos\phi \cdot \eta'' \qquad\qquad - [W_{\zeta\eta} \cdot \eta' + W_{\zeta\xi} \cdot \xi' + W_{\zeta\zeta} \cdot \zeta'] = 0. \end{aligned}$$

Similarly, if we assume the existence and continuity of all the derivatives which are needed, we obtain the following sets of identities:

$$(\alpha'') \qquad \begin{aligned} &\eta^{iv} + 2\omega(\sin\phi \cdot \xi''' + \cos\phi \cdot \zeta''') \\ &\quad - [W_{\eta\eta} \cdot \eta'' + W_{\eta\xi} \cdot \xi'' + W_{\eta\zeta} \cdot \zeta'' + \alpha_1 \eta' + \beta_1 \xi' + \gamma_1 \zeta'] = 0, \\ &\xi^{iv} - 2\omega \sin\phi \cdot \eta''' \\ &\quad - [W_{\xi\eta} \cdot \eta'' + W_{\xi\xi} \cdot \xi'' + W_{\xi\zeta} \cdot \zeta'' + \alpha_2 \eta' + \beta_2 \xi' + \gamma_2 \zeta'] = 0, \\ &\zeta^{iv} - 2\omega \cos\phi \cdot \eta''' \\ &\quad - [W_{\zeta\eta} \cdot \eta'' + W_{\zeta\xi} \cdot \xi'' + W_{\zeta\zeta} \cdot \zeta'' + \alpha_3 \eta' + \beta_3 \xi' + \gamma_3 \zeta'] = 0, \end{aligned}$$

where α_i, β_i, γ_i, $i = 1, 2, 3$, stand for homogeneous linear expressions in η', ξ', ζ', the coefficients of which are third derivatives of W*,

* If $u = u(x, y, z)$, $u' = u_x \cdot x' + u_y \cdot y' + u_z z'$,

$$u'' = u_x \cdot x'' + u_y \cdot y'' + u_z \cdot z'' + (u_{xx} x' + u_{xy} y' + u_{xz} z')\, x' + (u_{yx} x' + u_{yy} y' + u_{yz} z')\, y' + (u_{zx} x' + u_{zy} y' + u_{zz} z')\, z'.$$

$$\eta^{v} + 2\omega\,(\sin\phi \cdot \xi^{iv} + \cos\phi \cdot \zeta^{iv})$$
$$- [W_{\eta\eta} \cdot \eta''' + W_{\eta\xi} \cdot \xi''' + W_{\eta\zeta} \cdot \zeta''' + \alpha_1' \eta' + \beta_1' \xi' + \gamma_1' \zeta'] = 0,$$

(α''') $$\xi^{v} - 2\omega \sin\phi \cdot \eta^{iv}$$
$$- [W_{\xi\eta} \cdot \eta''' + W_{\xi\xi} \cdot \xi''' + W_{\xi\zeta} \cdot \zeta''' + \alpha_2' \cdot \eta' + \beta_2' \xi' + \gamma_2' \zeta'] = 0,$$

$$\zeta^{v} - 2\omega \cos\phi \cdot \eta^{iv}$$
$$- [W_{\zeta\eta} \cdot \eta''' + W_{\zeta\xi} \cdot \xi''' + W_{\zeta\zeta} \cdot \zeta''' + \alpha_3' \eta' + \beta_3' \xi' + \gamma_3' \zeta'] = 0,$$

where $\alpha_i', \beta_i', \gamma_i', i = 1, 2, 3$, stand for polynomials in the $\eta', \xi', \zeta', \eta'', \xi'', \zeta''$ and the third and fourth derivatives of W.*

By condition (β) we already know that

$$\eta_0 = \xi_0 = \zeta_0 = 0, \qquad \eta_0' = \xi_0' = \zeta_0' = 0.$$

We also know that

$$W_\eta^0 = W_\xi^0 = 0.$$

Therefore, when $t = 0$, the identities (α) yield the relations

$$\eta_0'' = 0, \qquad \xi_0'' = 0, \qquad \zeta_0'' = W_\zeta^0;$$

the identities (α'), the relations

$$\eta_0''' = -2\omega \cos\phi \cdot W_\zeta^0, \qquad \xi_0''' = 0, \quad \zeta_0''' = 0;$$

the identities (α''), the relations

$$\eta_0^{iv} = W_\zeta^0 W_{\eta\zeta}^0, \qquad \xi_0^{iv} = W_\zeta^0 (W_{\xi\zeta}^0 - 4\omega^2 \sin\phi \cos\phi),$$
$$\zeta_0^{iv} = W_\zeta^0 (W_{\zeta\zeta}^0 - 4\omega^2 \cos^2\phi);$$

and the identities (α'''), the relations

$$\eta_0^{v} = -2\omega W_\zeta^0 [\sin\phi \cdot W_{\xi\zeta}^0 + \cos\phi\,(W_{\zeta\zeta}^0 + W_{\eta\eta}^0 - 4\omega^2)],$$
$$\xi_0^{v} = +2\omega W_\zeta^0 [\sin\phi \cdot W_{\eta\zeta}^0 - \cos\phi \cdot W_{\xi\eta}^0],$$
$$\zeta_0^{v} = +2\omega W_\zeta^0 \cos\phi [W_{\eta\zeta}^0 - W_{\zeta\eta}^0] = 0.$$

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and

$$u''' = u_x x''' + u_y y''' + u_z z''' + 3[(u_{xx} x'' + u_{xy} y'' + u_{xz} z'')\,x' + (u_{xy} x'' + u_{yy} y'' + u_{yz} z'')\,y' + (u_{xz} x'' + u_{yz} y'' + u_{zz} z'')\,z']$$
$$+ \left\{ \begin{array}{c} (u_{xxx} x' + u_{xxy} y' + u_{xxz} z')\,x' \\ + (u_{xyx} x' + u_{xyy} y' + u_{xyz} z')\,y' \\ + (u_{xzx} x' + u_{xzy} y' + u_{xzz} z')\,z' \end{array} \right\} x' + \left\{ \begin{array}{c} (u_{yxx} x' + u_{yxy} y' + u_{yxz} z')\,x' \\ + (u_{yyx} x' + u_{yyy} y' + u_{yyz} z')\,y' \\ + (u_{yzx} x' + u_{yzy} y' + u_{yzz} z')\,z' \end{array} \right\} y'$$
$$+ \left\{ \begin{array}{c} (u_{zxx} x' + u_{zxy} y' + u_{zxz} z')\,x' \\ + (u_{zyx} x' + u_{zyy} y' + u_{zyz} z')\,y' \\ + (u_{zzx} x' + u_{zzy} y' + u_{zzz} z')\,z' \end{array} \right\} z'.$$

Let us now assume that the conditions are satisfied under which the set of differential equations (5) or (α) have a solution of the form

$$\begin{aligned}\eta &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots,\\ \xi &= b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + \cdots,\\ \zeta &= c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \cdots,\end{aligned} \tag{10}$$

in the neighborhood of the point P_0. It then follows, from the preceding work, that for the initial conditions (6) or (β),

$$\begin{aligned}
&a_0 = \eta_0 = 0, \quad a_1 = \eta_0' = 0, \quad a_2 = \tfrac{1}{2}\eta_0'' = 0,\\
&b_0 = \xi_0 = 0, \quad b_1 = \xi_0' = 0, \quad b_2 = \tfrac{1}{2}\xi_0'' = 0,\\
&c_0 = \zeta_0 = 0, \quad c_1 = \zeta_0' = 0, \quad c_2 = \tfrac{1}{2}\zeta_0'' = \tfrac{1}{2}W_\zeta^0;\\
&a_3 = \tfrac{1}{6}\eta_0''' = -\tfrac{1}{3}\omega\cos\phi\cdot W_\zeta^0,\\
&b_3 = \tfrac{1}{6}\xi_0''' = 0,\\
&c_3 = \tfrac{1}{6}\zeta_0''' = 0;\\
&a_4 = \tfrac{1}{24}\eta_0^{iv} = \tfrac{1}{24}W_\zeta^0 W_{\eta\zeta}^0,\\
&b_4 = \tfrac{1}{24}\xi_0^{iv} = \tfrac{1}{24}W_\zeta^0(W_{\xi\zeta}^0 - 4\omega^2\sin\phi\cos\phi),\\
&c_4 = \tfrac{1}{24}\zeta_0^{iv} = \tfrac{1}{24}W_\zeta^0(W_{\zeta\zeta}^0 - 4\omega^2\cos^2\phi);\\
&a_5 = \tfrac{1}{120}\eta_0^{v} = -\tfrac{1}{60}\omega W_\zeta^0[\sin\phi\cdot W_{\xi\zeta}^0 + \cos\phi\,(W_{\zeta\zeta}^0 + W_{\eta\eta}^0 - 4\omega^2)],\\
&b_5 = \tfrac{1}{120}\xi_0^{v} = \tfrac{1}{60}\omega W_\zeta^0[\sin\phi\cdot W_{\eta\zeta}^0 - \cos\phi\, W_{\xi\eta}^0],\\
&c_5 = \tfrac{1}{120}\zeta_0^{v} = 0.
\end{aligned} \tag{11}$$

The equations (10), for which the constants are given by the relations (11), are then the equations of the curve c referred to the cardinal axes at P_0.

§ 6. *The plumb-bob locus referred to the cardinal axes of* P_0. This curve has already been defined as the locus of plumb-bobs of all plumb-lines which are supported at the initial point P_0, of the falling particle. It is therefore the locus of the feet of perpendiculars dropped from P_0 ($\eta = 0$, $\xi = 0$, $\zeta = 0$) to the level (equipotential) surfaces $W = \text{const.}$ Its equations are easily seen to be

$$\frac{\eta}{\dfrac{\partial W}{\partial \eta}} = \frac{\xi}{\dfrac{\partial W}{\partial \xi}} = \frac{\zeta}{\dfrac{\partial W}{\partial \zeta}}. \tag{12}$$

Now let us solve these equations in the form

$$\eta = \alpha_1\zeta + \alpha_2\zeta^2 + \cdots, \qquad \xi = \beta_1\zeta + \beta_2\zeta^2 + \cdots.$$

In order to do this let us write the equations (12) in the form:

$$F_i(\eta, \xi, \zeta) = 0, \qquad (i = 1, 2)$$

where

$$F_1 = \begin{vmatrix} \xi & \zeta \\ W_\xi & W_\zeta \end{vmatrix} \quad \text{and} \quad F_2 = \begin{vmatrix} \zeta & \eta \\ W_\zeta & W_\eta \end{vmatrix}.$$

The derivatives $d\eta/d\zeta$, $d\xi/d\zeta$, $d^2\eta/d\zeta^2$, $d^2\xi/d\zeta^2$ are expressible in terms of the derivatives of F_i, $i = 1, 2$,* and the derivatives of F_i are expressible in terms of the derivatives of W.† The coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ have then the following values:

$$\alpha_1 = \left(\frac{d\eta}{d\zeta}\right)_0 = 0, \qquad \beta_1 = \left(\frac{d\xi}{d\zeta}\right)_0 = 0,$$

$$\alpha_2 = \frac{1}{2}\left(\frac{d^2\eta}{d\zeta^2}\right)_0 = \frac{W^0_{\eta\zeta}}{W^0_\zeta}, \qquad \beta_2 = \frac{W^0_{\xi\zeta}}{W^0_\zeta}.$$

Therefore the plumb-bob locus may be represented by the equations

$$\eta = \frac{W^0_{\eta\zeta}}{W^0_\zeta}\zeta^2 + \cdots, \qquad \xi = \frac{W^0_{\xi\zeta}}{W^0_\zeta}\zeta^2 + \cdots. \tag{13}$$

§ 7. *The curve c referred to axes of origin P_0 and directions which are parallel to the cardinal directions at the plumb-bob of a plumb-line which is supported at P_0.* Let us denote by R (Fig. 3) the point at which the plumb-bob of a

* The relations are the following:

$$\frac{\partial F_i}{\partial \eta}\frac{d\eta}{d\zeta} + \frac{\partial F_i}{\partial \xi}\frac{d\xi}{d\zeta} + \frac{\partial F_i}{\partial \zeta} = 0 \qquad (i = 1, 2),$$

$$\frac{\partial F_i}{\partial \eta}\frac{d^2\eta}{d\zeta^2} + \frac{d\eta}{d\zeta}\left[\frac{\partial^2 F_i}{\partial \eta \partial \xi}\frac{d\xi}{d\zeta} + \frac{\partial^2 F_i}{\partial \eta^2}\frac{d\eta}{d\zeta} + \frac{\partial^2 F_i}{\partial \zeta \partial \eta}\right] + \frac{\partial^2 F_i}{\partial \zeta \partial \eta}\frac{d\eta}{d\zeta} + \frac{\partial^2 F_i}{\partial \zeta^2} + \frac{\partial F_i}{\partial \xi}\frac{d^2\xi}{d\zeta^2}$$
$$+ \frac{d\xi}{d\zeta}\left[\frac{\partial^2 F_i}{\partial \xi^2}\frac{d\xi}{d\zeta} + \frac{\partial^2 F_i}{\partial \xi \partial \eta}\frac{d\eta}{d\zeta} + \frac{\partial^2 F_i}{\partial \zeta \partial \xi}\right] + \frac{\partial^2 F_i}{\partial \zeta \partial \xi}\frac{d\xi}{d\zeta} = 0 \quad (i = 1, 2).$$

† The relations which are needed are the following:

$$\frac{\partial F_1}{\partial \eta} = \begin{vmatrix} \xi & \zeta \\ W_{\xi\eta} & W_{\zeta\eta} \end{vmatrix}, \quad \frac{\partial F_1}{\partial \xi} = \begin{vmatrix} \xi & \zeta \\ W_{\xi\xi} & W_{\zeta\xi} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ W_\xi & W_\zeta \end{vmatrix}, \quad \frac{\partial F_1}{\partial \zeta} = \begin{vmatrix} \xi & \zeta \\ W_{\xi\zeta} & W_{\zeta\zeta} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ W_\xi & W_\zeta \end{vmatrix},$$

$$\frac{\partial F_2}{\partial \eta} = \begin{vmatrix} \zeta & \eta \\ W_{\zeta\eta} & W_{\eta\eta} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ W_\zeta & W_\eta \end{vmatrix}, \quad \frac{\partial F_2}{\partial \xi} = \begin{vmatrix} \zeta & \eta \\ W_{\zeta\xi} & W_{\eta\xi} \end{vmatrix}, \quad \frac{\partial F_2}{\partial \zeta} = \begin{vmatrix} \zeta & \eta \\ W_{\zeta\zeta} & W_{\eta\zeta} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ W_\zeta & W_\eta \end{vmatrix},$$

$$\frac{\partial^2 F_1}{\partial \zeta^2} = \begin{vmatrix} \xi & \zeta \\ W_{\xi\zeta\zeta} & W_{\zeta\zeta\zeta} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ W_{\xi\zeta} & W_{\zeta\zeta} \end{vmatrix} - W_{\xi\zeta},$$

$$\frac{\partial^2 F_2}{\partial \zeta^2} = \begin{vmatrix} \zeta & \eta \\ W_{\zeta\zeta\zeta} & W_{\eta\zeta\zeta} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ W_{\zeta\zeta} & W_{\eta\zeta} \end{vmatrix} + W_{\eta\zeta};$$

$$\therefore \left(\frac{\partial F_1}{\partial \eta}\right)_0 = 0, \quad \left(\frac{\partial F_1}{\partial \xi}\right)_0 = W^0_\zeta, \quad \left(\frac{\partial F_1}{\partial \zeta}\right)_0 = -W^0_\xi = 0, \quad \frac{\partial^2 F_1}{\partial \zeta^2} = -2W^0_{\xi\zeta},$$

$$\left(\frac{\partial F_2}{\partial \eta}\right)_0 = -W^0_\zeta, \quad \left(\frac{\partial F_2}{\partial \xi}\right)_0 = 0, \quad \left(\frac{\partial F_2}{\partial \zeta}\right)_0 = W^0_\eta = 0, \quad \frac{\partial^2 F_2}{\partial \zeta^2} = 2W^0_{\eta\zeta}.$$

plumb-line which is supported at P_0 is in equilibrium under the tension of the plumb-line and the force of the field whose potential function is represented by W. The line RP_0 is then the vertical at R. Let us denote by RE_R and RS_R the easterly and the southerly directions at R. Now let us draw through P_0 the axes $P_0\,\bar{\eta}$, $P_0\,\bar{\xi}$, $P_0\,\bar{\zeta}$ parallel to RE_R, RS_R and RP_0 respectively ($P_0\,\bar{\zeta}$ is the continuation of the plumb-line RP_0). Let us denote by $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$ the directional angles of the axes $P_0 - \bar{\eta}$, $\bar{\xi}$, $\bar{\zeta}$ with respect to the cardinal axes $P_0 - \eta$, ξ, ζ. Then

$$\begin{aligned}
\bar{\eta} &= \cos\alpha_1 \cdot \eta + \cos\beta_1 \cdot \xi + \cos\gamma_1 \cdot \zeta, \\
\bar{\xi} &= \cos\alpha_2 \cdot \eta + \cos\beta_2 \cdot \xi + \cos\gamma_2 \cdot \zeta, \\
\bar{\zeta} &= \cos\alpha_3 \cdot \eta + \cos\beta_3 \cdot \xi + \cos\gamma_3 \cdot \zeta.
\end{aligned} \tag{14}$$

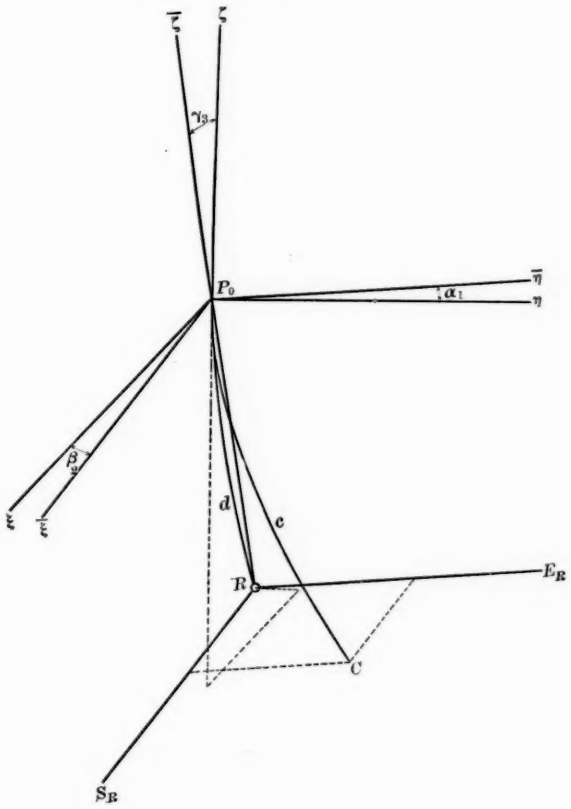

FIG. 3.

In order to obtain expressions for these directional cosines let us recall that, at a general point P, (1) the vertical is the normal to the level (equipotential) surface which passes through P, (2) the east-and-west line is the intersection of the horizontal plane and the plane (z = const.) which passes through P and is perpendicular to the axis of rotation of the earth, (3) the north-and-south line is the common perpendicular to the vertical and the east-and-west line. Therefore, at a general point P, (η, ξ, ζ), (1) the directional cosines * of the vertical are proportional to

$$W_\eta, \quad W_\xi, \quad W_\zeta;$$

(2) those of the east-and-west line are proportional to

$$H, \quad \Xi, \quad Z,$$

where

$$H = \begin{vmatrix} W_\xi & W_\zeta \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \zeta} \end{vmatrix} = \sin\phi \cdot W_\xi + \cos\phi \cdot W_\zeta,$$

$$\Xi = \begin{vmatrix} W_\zeta & W_\eta \\ \frac{\partial z}{\partial \zeta} & \frac{\partial z}{\partial \eta} \end{vmatrix} = -\sin\phi \cdot W_\eta,$$

$$Z = \begin{vmatrix} W_\eta & W_\xi \\ \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \xi} \end{vmatrix} = -\cos\phi \cdot W_\eta;$$

and (3) those of the north-and-south line are proportional to

$$\begin{vmatrix} W_\xi & W_\zeta \\ \Xi & Z \end{vmatrix}, \quad \begin{vmatrix} W_\zeta & W_\eta \\ Z & H \end{vmatrix}, \quad \begin{vmatrix} W_\eta & W_\xi \\ H & \Xi \end{vmatrix}.$$

If the point P is not too far from P_0, we may write without appreciable error,

$$\begin{aligned} W_\eta &= W^0_{\eta\eta} \cdot \eta + W^0_{\eta\xi} \cdot \xi + W^0_{\eta\zeta} \cdot \zeta, \\ W_\xi &= W^0_{\xi\eta} \cdot \eta + W^0_{\xi\xi} \cdot \xi + W^0_{\xi\zeta} \cdot \zeta, \\ W_\zeta &= W^0_\zeta + W^0_{\zeta\eta} \cdot \eta + W^0_{\zeta\xi} \cdot \xi + W^0_{\zeta\zeta} \cdot \zeta. \end{aligned}$$

Therefore, for the point R, whose coordinates are

$$\eta = \frac{W^0_{\eta\zeta}}{W^0_\zeta}\zeta^2, \qquad \xi = \frac{W^0_{\xi\zeta}}{W^0_\zeta}\zeta^2, \qquad \zeta = \zeta,$$

we may write

$$W^R_\eta = W^0_{\eta\zeta} \cdot \zeta, \qquad W^R_\xi = W^0_{\xi\zeta} \cdot \zeta, \qquad W^R_\zeta = W^0_\zeta + W^0_{\zeta\zeta} \cdot \zeta,$$

* With respect to the cardinal directions of P_0.

and hence

$$\cos\alpha_3 : \cos\beta_3 : \cos\gamma_3 = W^0_{\eta\zeta}\cdot\zeta : W^0_{\xi\zeta}\cdot\zeta : W^0_\zeta + W^0_{\zeta\zeta}\cdot\zeta,$$

$$\cos\alpha_1 : \cos\beta_1 : \cos\gamma_1$$

$$= \cos\phi\cdot W^0_\zeta + (\sin\phi\cdot W^0_{\xi\zeta} + \cos\phi\cdot W^0_{\zeta\zeta})\,\zeta : -\sin\phi\cdot W^0_{\eta\zeta}\cdot\zeta : -\cos\phi\cdot W^0_{\eta\zeta}\cdot\zeta,$$

$$\cos\alpha_2 : \cos\beta_2 : \cos\gamma_2$$

$$= \sin\phi\cdot W^0_\zeta\cdot W^0_{\eta\zeta}\cdot\zeta : \cos\phi\,(W^0_\zeta)^2 + W^0_\zeta(\sin\phi\cdot W^0_{\xi\zeta} + 2\cos\phi\cdot W^0_{\zeta\zeta})\,\zeta : -\cos\phi\cdot W^0_\zeta\cdot W^0_{\xi\zeta}\cdot\zeta,$$

for terms of order not greater than the first. In order to get the cosines themselves, we observe that *

$$\left((W^0_{\eta\zeta}\cdot\zeta)^2 + (W^0_{\xi\zeta}\cdot\zeta)^2 + (W^0_\zeta + W^0_{\zeta\zeta}\,\zeta)^2\right)^{-\frac{1}{2}} = \frac{1}{W^0_\xi}\left(1 - \frac{W^0_{\zeta\zeta}}{W^0_\zeta}\zeta + \cdots\right),$$

$$\left(\{\cos.\,\phi\cdot W^0_\zeta + (\sin\phi\cdot W^0_{\xi\zeta} + \cos\phi\cdot W^0_{\zeta\zeta})\,\zeta\}^2 + (-\sin\phi\cdot W^0_{\eta\zeta}\cdot\zeta)^2 + (-\cos\phi\cdot W^0_{\eta\zeta}\cdot\zeta)^2\right)^{-\frac{1}{2}} = \frac{1}{\cos\phi\cdot W^0_\zeta}\left(1 - \frac{\sin\phi\cdot W^0_{\xi\zeta} + \cos\phi\cdot W^0_{\zeta\zeta}}{\cos\phi\cdot W^0_\zeta}\zeta + \cdots\right),$$

$$\left((\sin\phi\cdot W^0_\zeta\cdot W^0_{\eta\zeta}\cdot\zeta)^2 + \{\cos\phi\cdot(W^0_\zeta)^2 + W^0_\zeta(\sin\phi\cdot W^0_{\xi\zeta} + 2\cos\phi\cdot W^0_{\zeta\zeta})\,\zeta\}^2 + (-\cos\phi\cdot W^0_\zeta\cdot W^0_{\xi\zeta}\cdot\zeta)^2\right)^{-\frac{1}{2}}$$

$$= \frac{1}{\cos\phi\cdot(W^0_\zeta)^2}\left(1 - \frac{\sin\phi\cdot W^0_{\xi\zeta} + 2\cos\phi\cdot W^0_{\zeta\zeta}}{\cos\phi\cdot W^0_\zeta}\zeta + \cdots\right).$$

Therefore

$$\cos\alpha_3 = \frac{W^0_{\eta\zeta}}{W^0_\zeta}\zeta, \qquad \cos\beta_3 = \frac{W^0_{\xi\zeta}}{W^0_\zeta}\zeta, \qquad \cos\gamma_3 = 1,$$

$$(15)\quad \cos\alpha_1 = 1, \qquad \cos\beta_1 = -\tan\phi\,\frac{W^0_{\eta\zeta}}{W^0_\zeta}\zeta, \qquad \cos\gamma_1 = -\frac{W^0_{\eta\zeta}}{W^0_\zeta}\cdot\zeta,$$

$$\cos\alpha_2 = \tan\phi\cdot\frac{W^0_{\eta\zeta}}{W^0_\zeta}\cdot\zeta, \qquad \cos\beta_2 = 1, \qquad \cos\gamma_2 = -\frac{W^0_{\xi\zeta}}{W^0_\zeta}\cdot\zeta,$$

for terms of order not higher than the first. In virtue of the values (15), equations (14) assume the form:

$$\bar\eta = \eta - \tan\phi\,\frac{W^0_{\eta\zeta}}{W^0_\zeta}\xi\zeta - \frac{W^0_{\eta\zeta}}{W^0_\zeta}\zeta^2,$$

$$(16)\qquad \bar\xi = \tan\phi\cdot\frac{W^0_{\eta\zeta}}{W^0_\zeta}\eta\zeta + \xi - \frac{W^0_{\xi\zeta}}{W^0}\zeta^2,$$

$$\bar\zeta = \frac{W^0_{\eta\zeta}}{W^0_\zeta}\eta\zeta + \frac{W^0_{\xi\zeta}}{W^0_\zeta}\xi\zeta + \zeta,$$

* $\dfrac{1}{\sqrt{A + B\zeta + C\zeta^2 + \cdots}} = \dfrac{1}{A^{\frac{1}{2}}}\left(1 - \dfrac{1}{2}\dfrac{B}{A}\zeta + \dfrac{3B^2 - 4AC}{8A^2}\zeta^2 + \cdots\right).$

in which we must now substitute for η, ξ, ζ the values in t which are given by equations (10). Doing this, we find *

$$\bar{\eta} = -\tfrac{1}{3}\omega\cos\phi\cdot W^0_\zeta\cdot t^3 - \tfrac{5}{24}W^0_\zeta\cdot W^0_{\eta\zeta}\cdot t^4 - \tfrac{1}{60}\omega W^0_\zeta(\sin\phi\cdot W^0_{\xi\zeta} + \cos\phi\,(W^0_{\zeta\zeta} + W^{0\eta}_{\eta} - 4\omega^2))t^5,$$

$$(17)\quad \bar{\xi} = -(\tfrac{1}{6}\omega^2\sin\phi\cos\phi + \tfrac{5}{24}W^0_{\xi\zeta})W^0_\zeta\cdot t^4 - \tfrac{1}{60}\omega W^0_\zeta(9\sin\phi\,W^0_{\eta\zeta} + \cos\phi\,W^0_{\xi\eta})t^5,$$

$$\bar{\zeta} = \tfrac{1}{2}W^0_\zeta\cdot t^2 + \tfrac{1}{24}(W^0_{\zeta\zeta} - 4\omega^2\cos^2\phi)W^0_\zeta\cdot t^4 - \tfrac{1}{6}\omega\cos\phi\,W^0_\zeta W^0_{\eta\zeta}t^5.$$

Equations (17) are the equations of the curve c referred to the axes $P_0 - \bar{\eta}, \bar{\xi}, \bar{\zeta}$.

If we put $-\bar{\zeta} = h = R\,P_0$, where h is the height through which the particle falls, then the corresponding values of $\bar{\eta}$ and $\bar{\xi}$ are the *easterly* and the *southerly deviations* which correspond to h. These, and *not* the quantities η, ξ, are the ones which are measured in experiments.

WASHINGTON UNIVERSITY,
ST. LOUIS, MO., *April*, 1912.

* We can not carry these developments beyond t^5 unless we carry the developments (15) beyond ζ.

ON APPROXIMATION BY TRIGONOMETRIC SUMS AND POLYNOMIALS*

BY

DUNHAM JACKSON

The chief purpose of this paper, carried out in its second part, is the determination of numerical limits for certain constants, hitherto undetermined, which figure in the principal theorems of the first two parts (Abschnitte) of the author's thesis,† concerning the degree of approximation to a given continuous function $f(x)$ that can be attained uniformly in an interval by a polynomial of the nth degree in x, or for all values of x by a trigonometric sum of the nth order. By a "trigonometric sum of the nth order at most" is meant an expression of the form

$$a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx,$$

where the coefficients a_i, b_i, are constants. In the first part of the paper, occasion is taken to present considerably simplified proofs of those theorems themselves, with some modifications in statement.‡ The simplification depends on a recognition of the fact that it is possible to treat the trigonometric case directly and deduce thence the results in the polynomial case; the opposite order of treatment, employed in the thesis, requires a considerably longer discussion. The reader is referred to the introduction of the latter for a review of the literature of the subject, down to the spring of 1911. As far as logical development is concerned, the present paper may be read quite independently of the thesis, though, allowing for the fundamental change in arrangement, the first part of this article employs to a considerable extent the methods of the sections of the thesis referred to, and various matters of detail are more fully presented there than here.

In addition to the theorems thus based directly on the thesis, the article

* Presented to the Society, February 24 and April 27, 1912.

† *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung.* Dissertation, Göttingen, 1911. Referred to hereafter as *Thesis.*

‡ See footnotes attached to the several theorems.

contains two others, the fifth and the tenth, the status of which is explained where they are taken up in detail. In connection with the latter, the answer is found to a question raised by FEJÉR* concerning the quantities, met with in the theory of Fourier's series, which he calls "LEBESGUE's constants." Apart from this point, the substance of the article is indicated by the ten theorems, which are self-explanatory.

I. GENERAL THEOREMS

THEOREM I.† *There exists an absolute numerical constant K_1 having the following property: If $f(x)$ is a real function of the real variable x, of period 2π, which everywhere satisfies the Lipschitz condition*

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|, \tag{1}$$

λ being a constant, then there exists for every positive integral value of n a trigonometric sum $T_n(x)$, of the nth order at most, such that for all values of x

$$|f(x) - T_n(x)| \leqq \frac{K_1 \lambda}{n}.$$

We suppose the function $f(x)$ given. We define an approximating function $I_m(x)$ as follows:

$$I_m(x) = h_m \int_{-\pi/2}^{\pi/2} f(x + 2u) \left[\frac{\sin mu}{m \sin u}\right]^4 du,$$

where m is any positive integer, and h_m is defined by the equation

$$\frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^4 du. \tag{2}$$

We note first that $I_m(x)$ is a trigonometric sum in x, of order $2(m-1)$ at most. For by the substitution $x + 2u = v$ we find

$$I_m(x) = \frac{h_m}{2} \int_{x-\pi}^{x+\pi} f(v) \left[\frac{\sin m \frac{v-x}{2}}{m \sin \frac{v-x}{2}}\right]^4 dv,$$

or, since $f(v)$ and the fourth power (in fact any even power) of the quantity in brackets are both unaltered by a change of 2π in the value of v,

*Since the completion of this paper, an article by GRONWALL, Mathematische Annalen, vol. 72 (1912), pp. 244–261, has come to hand, in which the same problem is solved.

† *Thesis*, Satz VI, and remark at the top of page 46. In the case of this theorem the proof here given is not essentially different from the old one.

$$I_m(x) = \frac{h_m}{2}\int_{-\pi}^{\pi} f(v)\left[\frac{\sin m\dfrac{v-x}{2}}{m\sin\dfrac{v-x}{2}}\right]^4 dv.$$

If we show that the fourth power in the integrand is a trigonometric sum in $(v-x)$, of order not higher than $2(m-1)$, it will follow at once that $I_m(x)$ is such a sum in x. Now if m is odd, setting $v-x=t$ for brevity, we have the well-known identity

$$\frac{\sin(mt/2)}{\sin(t/2)} = 2\left(\tfrac{1}{2} + \cos t + \cos 2t + \cdots + \cos\tfrac{1}{2}(m-1)t\right),$$

the right-hand member of which is a trigonometric sum of order $\frac{1}{2}(m-1)$ in t; its fourth power is then a trigonometric sum of order $2(m-1)$ in t. On the other hand, if m is even, $=2m'$, we may write

$$\frac{\sin(mt/2)}{\sin(t/2)} = \frac{\sin m't}{\sin t}\cdot\frac{\sin t}{\sin(t/2)}.$$

It is readily proved by induction that $\sin m't/\sin t$ is a trigonometric sum in t of order $m'-1=\frac{1}{2}(m-2)$; and while $\sin t/\sin(t/2)=2\cos(t/2)$ itself is not a trigonometric sum in t, its square is such a sum, of order 1, and its fourth power is a sum of order 2. It follows that the function in the integrand is once more a trigonometric sum of order $2(m-1)$.

Having established this fact, we proceed to justify the use of the word "approximating" in connection with the function $I_m(x)$. Multiplying by $h_m f(x)$ the equation (2) which defines h_m, we obtain

$$f(x) = h_m\int_{-\pi/2}^{\pi/2} f(x)\left[\frac{\sin mu}{m\sin u}\right]^4 du.$$

Hence

$$I_m(x) - f(x) = h_m\int_{-\pi/2}^{\pi/2}[f(x+2u)-f(x)]\left[\frac{\sin mu}{m\sin u}\right]^4 du.$$

By the hypothesis expressed in (1),

$$|f(x+2u)-f(x)| \leqq 2\lambda|u|,$$

from which follows:

$$|I_m(x)-f(x)| \leqq 2\lambda h_m\int_{-\pi/2}^{\pi/2}|u|\left[\frac{\sin mu}{m\sin u}\right]^4 du = 4\lambda h_m\int_0^{\pi/2} u\left[\frac{\sin mu}{m\sin u}\right]^4 du,$$

or, if we substitute the value of h_m from (2) and make use of the fact that the

integrand there also is an even function,

$$(3) \qquad |I_m(x) - f(x)| \leqq 2\lambda \frac{\int_0^{\pi/2} u\left[\frac{\sin mu}{m \sin u}\right]^4 du}{\int_0^{\pi/2}\left[\frac{\sin mu}{m \sin u}\right]^4 du}.$$

Let us consider first the denominator of this fraction.

$$\int_0^{\pi/2}\left[\frac{\sin mu}{m \sin u}\right]^4 du > \int_0^{\pi/2}\left[\frac{\sin mu}{mu}\right]^4 du$$
$$= \frac{1}{m}\int_0^{m\pi/2} \frac{\sin^4 u}{u^4} du \geqq \frac{1}{m}\int_0^{\pi/2} \frac{\sin^4 u}{u^4} du,$$

and in this fourth member the integral factor is independent of m.

In the numerator, we make use of the fact that $(\sin u)/u \geqq 2/\pi$ in the interval of integration, and see that

$$\int_0^{\pi/2} u\left[\frac{\sin mu}{m \sin u}\right]^4 du \leqq \left(\frac{\pi}{2}\right)^4 \int_0^{\pi/2} u\left[\frac{\sin mu}{mu}\right]^4 du = \frac{1}{m^2}\left(\frac{\pi}{2}\right)^4 \int_0^{m\pi/2} \frac{\sin^4 u}{u^3} du$$
$$< \frac{1}{m^2}\left(\frac{\pi}{2}\right)^4 \int_0^{\infty} \frac{\sin^4 u}{u^3} du,$$

where m appears in the final expression only in the factor $1/m^2$.

By combination of these inequalities, it appears at once that the quantity $|I_m(x) - f(x)|$ does not exceed a constant multiple of λ/m, and Theorem I follows at once. To be sure, the order of the trigonometric sum $I_m(x)$ is not m, in general, but $2(m-1)$; and this takes on only even values, however m may be chosen. But the mere naming of these discrepancies suffices to show that they are inessential; if n is any positive integer, it will be possible to choose an m so that the function $I_m(x)$ will serve as the sum $T_n(x)$ required.*

Before going further, let us notice that in the application of Theorem I, or any similar theorem, if $f(x)$ is an even function, it may be assumed that the sums $T_n(x)$ have the same property. It is readily seen that the particular functions $I_m(x)$ used above will be even if $f(x)$ is even, but it is true more generally that if $f(x)$ is any even continuous function and $T_n(x)$ any trigonometric sum of the nth order whatever, and ϵ is a number such that

$$|f(x) - T_n(x)| \leqq \epsilon,$$

* For greater detail on this point, see the remark preceding formula (17) in the proof of Theorem VI below.

for all values of x, then it is true also that

$$(4)\quad \left| f(x) - \frac{T_n(x)+T_n(-x)}{2} \right| = \left| \frac{f(x)-T_n(x)}{2} + \frac{f(-x)-T_n(-x)}{2} \right| \leqq \epsilon,$$

and $\frac{1}{2}(T_n(x) + T_n(-x))$ is obviously a trigonometric sum of the same order as $T_n(x)$ or lower, which is at the same time an even function.

This well-known fact being recalled, let us drop the assumption that $f(x)$ is periodic, and suppose merely that it satisfies the Lipschitz condition (1) in the closed interval $-1 \leqq x \leqq 1$. Then $g(x) = f(\cos x)$ is a function of x defined for all real values of x. Moreover, it satisfies everywhere the condition (1). For

$$|g(x_2) - g(x_1)| = |f(\cos x_2) - f(\cos x_1)| \leqq \lambda |\cos x_2 - \cos x_1| \leqq \lambda |x_2 - x_1|.$$

By Theorem I, then, it is possible to find for each value of n a trigonometric sum $T_n(x)$, of order n or less, such that the inequality

$$|g(x) - T_n(x)| \leqq \frac{K_1 \lambda}{n}$$

is everywhere satisfied. As $g(x)$ is a function of $\cos x$, it is an even function of x, and hence we may assume that $T_n(x)$ is an even function of x. Then $T_n(x)$ has no sine terms, but involves only cosines, and so is a polynomial in $\cos x$, of degree not higher than the nth,

$$T_n(x) = \Pi_n(\cos x).$$

We have thus a representation of $f(\cos x)$ by the polynomial $\Pi_n(\cos x)$, or of $f(x)$ by $\Pi_n(x)$, for all values of the argument in the interval $(-1, 1)$, with a maximum error not greater than $K_1 \lambda / n$.

If the function $f(x)$ were given in some other interval than $(-1, 1)$, we should reduce that interval to this by a linear transformation of the form $x' = Ax + B$. The transformed function would be represented by a polynomial of the same degree, with the same error, as the original one. The transformed function would still satisfy a Lipschitz condition, the coefficient λ in this condition being altered in the inverse ratio of the lengths of the intervals. For example, the problem of representing a function which satisfies a Lipschitz condition with coefficient λ in an interval of length 1 would be reduced to that of representing a function which satisfies a Lipschitz condition with coefficient $\frac{1}{2}\lambda$ in the interval $(-1, 1)$ of length 2. We are led to the general statement which follows:

THEOREM II.* *There exists an absolute numerical constant L_1 having the*

* *Thesis*, Satz IV. The concluding statement, as to the relative values of K_1 and L_1, was not contained in the thesis.

following property: If $f(x)$ is a function of x which satisfies the Lipschitz condition (1) *throughout a closed interval $a \leqq x \leqq b$, of length l, then there exists for every positive integral value of n a polynomial $\Pi_n(x)$, of degree n at most, such that, for all values of x in the interval,*

$$|f(x) - \Pi_n(x)| \leqq \frac{L_1 l \lambda}{n}.$$

In particular, the constant $L_1 = \frac{1}{2}K_1$ has this property, if K_1 is a constant possessing the property described in Theorem I.

A generalization of the first theorem is

THEOREM III.* *For each positive integral value of k there exists a constant K_k having the following property: If $f(x)$ is a function of period 2π possessing a $(k-1)$th derivative which everywhere satisfies the Lipschitz condition*

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \leqq \lambda |x_2 - x_1|, \tag{5}$$

λ being a constant, then there exists for every positive integral value of n a trigonometric sum $T_n(x)$, of the nth order at most, such that for all values of x

$$|f(x) - T_n(x)| \leqq \frac{K_k \lambda}{n^k}.$$

Let us begin by considering the case $k = 2$. Instead of the previous definition of $I_m(x)$ we adopt the following:

$$I_m(x) = \frac{h_m}{2} \int_{-\pi/2}^{\pi/2} [f(x+2u) + f(x-2u)] \left[\frac{\sin mu}{m \sin u}\right]^4 du;$$

h_m has the same meaning as before. Various details of the earlier work will be again applicable, and need not be repeated explicitly. The new $I_m(x)$ is still a trigonometric sum in x, of order $2(m-1)$ or lower. We find

$$I_m(x) - f(x) = \frac{h_m}{2} \int_{-\pi/2}^{\pi/2} [f(x+2u) - 2f(x) + f(x-2u)] \left[\frac{\sin mu}{m \sin u}\right]^4 du.$$

By reason of the hypothesis concerning the derivative of $f(x)$,

$$f(x+2u) = f(x) + 2uf'(x + \theta_1 \cdot 2u) = f(x) + 2uf'(x) + \theta_1' \cdot 4\lambda u^2,$$

$$f(x-2u) = f(x) - 2uf'(x + \theta_2 \cdot 2u) = f(x) - 2uf'(x) + \theta_2' \cdot 4\lambda u^2,$$

where θ_1 and θ_2 are between 0 and 1, and θ_1' and θ_2' between -1 and 1. Hence

$$|f(x+2u) - 2f(x) + f(x-2u)| \leqq 8\lambda u^2,$$

* *Thesis*, Satz VII. The present statement is however more precise than the old one. See also footnote under Theorem IV.

and

$$(6)\quad |I_m(x)-f(x)| \leqq 4\lambda h_m \int_{-\pi/2}^{\pi/2} u^2 \left[\frac{\sin mu}{m \sin u}\right]^4 du = 4\lambda \frac{\int_0^{\pi/2} u^2 \left[\frac{\sin mu}{m \sin u}\right]^4 du}{\int_0^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^4 du}.$$

We have seen that the denominator of this fraction remains greater than a constant multiple of $1/m$. As to the numerator,

$$\int_0^{\pi/2} u^2 \left[\frac{\sin mu}{m \sin u}\right]^4 du < \left(\frac{\pi}{2}\right)^4 \int_0^{\pi/2} u^2 \left[\frac{\sin mu}{mu}\right]^4 du < \frac{1}{m^3}\left(\frac{\pi}{2}\right)^4 \int_0^{\infty} \frac{\sin^4 u}{u^2} du.$$

Hence the quantity $|I_m(x) - f(x)|$ does not exceed a constant multiple of λ/m^2, and Theorem II for $k = 2$ follows immediately.

In constructing a proof which shall be applicable for any value of k, we shall make use of an approximating function $I_m(x)$ which does not reduce to that just employed when $k = 2$, but to an analogous expression which might have been used instead of that one.

First, we show that if i and κ are positive integers the function

$$J(x) = \int_{-\pi/2}^{\pi/2} f(x + 2iu) \left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du$$

is a trigonometric sum in x. Let us make the substitution

$$x + 2iu = v.$$

The integral $J(x)$ becomes

$$\frac{1}{2i}\int_{x-i\pi}^{x+i\pi} f(v) \left[\frac{\sin m\frac{v-x}{2i}}{m \sin \frac{v-x}{2i}}\right]^{2\kappa} dv,$$

which is the same thing as

$$\frac{1}{2i}\int_{-i\pi}^{i\pi} f(v) \left[\frac{\sin m\frac{v-x}{2i}}{m \sin \frac{v-x}{2i}}\right]^{2\kappa} dv,$$

because of the periodicity of the integrand. From this form we see by a process of reasoning used before that $J(x)$ is at any rate a trigonometric sum in x/i. Let us break it up into i integrals, and write

$$2iJ(x) = \int_{-i\pi}^{-(i-2)\pi} + \int_{-(i-2)\pi}^{-(i-4)\pi} + \cdots + \int_{(i-2)\pi}^{i\pi}.$$

If we replace x by $x + 2\pi$ in the last of these integrals, we get

$$\int_{(i-2)\pi}^{i\pi} f(v)\left[\frac{\sin m\dfrac{v-(x+2\pi)}{2i}}{m\sin\dfrac{v-(x+2\pi)}{2i}}\right]^{2\kappa} dv,$$

and this, if we set $v - 2\pi = v'$ and remember that $f(v') = f(v)$, is equal to

$$\int_{(i-4)\pi}^{(i-2)\pi} f(v')\left[\frac{\sin m\dfrac{v'-x}{2i}}{m\sin\dfrac{v'-x}{2i}}\right]^{2\kappa} dv';$$

that is, the substitution of $x + 2\pi$ for x carries over the last of the i integrals into the next preceding one. Similarly, it will carry each of the others, except the first, into the one before it. The first will go over into

$$\int_{-(i+2)\pi}^{-i\pi} f(v')\left[\frac{\sin m\dfrac{v'-x}{2i}}{m\sin\dfrac{v'-x}{2i}}\right]^{2\kappa} dv' = \int_{(i-2)\pi}^{i\pi} f(v'')\left[\frac{\sin m\dfrac{v''-x}{2i}}{m\sin\dfrac{v''-x}{2i}}\right]^{2\kappa} dv'',$$

$$v'' = v' + 2i\pi;$$

the first integral goes over precisely into the ith. Hence $J(x)$ has the period 2π, and the terms in its expansion which involve multiples of x/i other than whole multiples of x must vanish.* It is of order $\kappa(m-1)$ or less in x/i, accordingly of order not higher than $\kappa(m-1)/i$ in x.

With this information, let us choose κ, for any given value of k, as the smallest integer such that $2\kappa - k > 1$, so that κ is perfectly determinate when k is given, and set

$$I_m(x) = h_m \int_{-\pi/2}^{\pi/2} [\pm f(x + 2ku) \mp kf(x + 2(k-1)u) + \cdots + kf(x + 2u)]\left[\frac{\sin mu}{m\sin u}\right]^{2\kappa} du,$$

where the numerical coefficients of the terms in the first factor of the integrand are the binomial coefficients corresponding to the exponent k, and h_m is defined by the equation

$$\frac{1}{h_m} = \int_{-\pi/2}^{\pi/2}\left[\frac{\sin mu}{m\sin u}\right]^{2\kappa} du.$$

* This can be deduced without difficulty from the fact that a trigonometric sum can vanish identically only if all its coefficients are zero.

The function so defined is a trigonometric sum in x, of order $\kappa(m-1)$ at most.

The difference $I_m(x) - f(x)$ may be written in a form which differs from that of $I_m(x)$ only in having one more term, $-f(x)$, in the first factor of the integrand. If this factor is developed by Taylor's theorem, with the aid of (5), to terms of the kth degree in u, it is found that the terms of degree lower than the kth in the expansion combine so that their sum is identically zero, and the whole expression does not exceed a constant multiple of $\lambda|u|^k$ in absolute value. It will be seen on writing out the expansion at length that this statement will be justified if we can show that the function

$$S_{k,i}(t) = t^i - k(t+1)^i + \cdots + (-1)^{k-1} k(t+k-1)^i + (-1)^k (t+k)^i,$$

where the numerical coefficients are still the binomial coefficients for the exponent k, vanishes identically whenever $0 \leqq i \leqq k-1$, i being an integer; what is needed is only the fact that it vanishes when $t = 0$. As

$$\frac{d}{dt} S_{k,i}(t) \equiv i S_{k,i-1}(t),$$

$S_{k,i-1}(t)$ vanishes identically if $S_{k,i}(t)$ does, and it is sufficient to prove for each value of k that $S_{k,k-1}(t)$ is identically zero. The desired proof is obtained by induction. Suppose $S_{k-1,k-2}(t) \equiv 0$; this is true, for example, if $k = 2$. Then

$$\frac{d}{dt} S_{k-1,k-1}(t) \equiv (k-1) S_{k-1,k-2}(t) \equiv 0,$$

and hence $S_{k,k-1}(t)$, which is equal to $S_{k-1,k-1}(t) - S_{k-1,k-1}(t+1)$, is identically zero, and the induction is complete.* It follows that the absolute value of $I_m(x) - f(x)$ does not exceed

$$\lambda \frac{\int_0^{\pi/2} u^k \left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du}{\int_0^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du}$$

multiplied by a quantity which is dependent on k and on nothing else. For the denominator and the numerator of this fraction we have the inequalities

$$\int_0^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du > \int_0^{\pi/2} \left[\frac{\sin mu}{mu}\right]^{2\kappa} du \geqq \frac{1}{m} \int_0^{\pi/2} \frac{\sin^{2\kappa} u}{u^{2\kappa}} du;$$

$$\int_0^{\pi/2} u^k \left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du < \left(\frac{\pi}{2}\right)^{2\kappa} \int_0^{\pi/2} u^k \left[\frac{\sin mu}{mu}\right]^{2\kappa} du < \frac{1}{m^{k+1}} \left(\frac{\pi}{2}\right)^{2\kappa} \int_0^{\infty} \frac{\sin^{2\kappa} u}{u^{2\kappa-k}} du;$$

* Cf. *Thesis*, pp. 29–30, where a somewhat different proof is given.

and the last integral actually exists, since we have supposed $2\kappa - k > 1$. Hence $|I_m(x) - f(x)|$ does not exceed a constant multiple of λ/m^k, and by establishing a suitable relation between m and n the functions $I_m(x)$ may be made to serve the purpose of the sums $T_n(x)$ demanded by Theorem III.

A corresponding generalization of Theorem II is possible. Restricting ourselves at first to the case $k = 2$, let us suppose that $f(x)$ is a function of x defined in the interval $-1 \leqq x \leqq 1$, and possessing there a derivative $f'(x)$ which satisfies the condition

$$|f'(x_2) - f'(x_1)| \leqq \lambda |x_2 - x_1|. \tag{7}$$

Then $g(x) = f(\cos x)$ is defined for all values of x; it has everywhere a derivative

$$g'(x) = -\sin x \cdot f'(\cos x),$$

and this satisfies the condition

$$\begin{aligned} |g'(x_2) - g'(x_1)| &= |-\sin x_2 f' \cos(x_2) + \sin x_1 f'(\cos x_1)| \\ &= |-\sin x_2 [f'(\cos x_2) - f'(\cos x_1)] - f'(\cos x_1)(\sin x_2 - \sin x_1)| \\ &\leqq |f'(\cos x_2) - f'(\cos x_1)| + |f'(\cos x_1)|\,|x_2 - x_1|. \end{aligned} \tag{8}$$

As we may subtract a linear function from $f(x)$ without altering the problem of approximating to it by a polynomial of given positive degree, we may assume without loss of generality that $f'(0) = 0$; then it follows from condition (7) that throughout the interval of definition $|f'(x)| \leqq \lambda$. Applying (7) once more in the relation (8), we find

$$|g'(x_2) - g'(x_1)| \leqq 2\lambda |x_2 - x_1|.$$

Accordingly $g(x) = f(\cos x)$ can be approximately represented by a trigonometric sum of order n or less, and hence by a polynomial in $\cos x$ of degree n or less, for all values of x, with an error not exceeding $2K_2\lambda/n^2$, where K_2 is the constant of Theorem III for the case $k = 2$. That is, $f(x)$ can be so represented by a polynomial of not higher degree than the nth in x, throughout the interval $-1 \leqq x \leqq 1$.

If $f(x)$ satisfies the condition (7) in some other interval than this, and the given interval is reduced to this one by a linear transformation of the form $x' = Ax + B$, the condition (7) will be preserved, except that the coefficient λ will be altered in the inverse ratio of the squares of the lengths of the intervals. If the given interval were of length 1, for example, the Lipschitz condition after the transformation to the interval $(-1, 1)$ would have the coefficient $\frac{1}{4}\lambda$, and a representation would be obtained with a maximum error not exceeding $\frac{1}{2}K_2\lambda/n^2$.

The reasoning based on the hypothesis that $f(x)$ has a $(k-1)$th derivative satisfying (5) in a closed interval is similar. Suppose first that the interval is $(-1, 1)$. We consider again the function $g(x) = f(\cos x)$. The derivative $g^{(k-1)}(x)$ exists everywhere, and is given by a polynomial in the derivatives $f^{(i)}(\cos x)$, $i = 1, 2, \cdots, k-1$, and $\sin x$ and $\cos x$, which is linear in the derivatives of f. By subtracting a suitable polynomial of degree $k-1$ from $f(x)$ at the beginning, we can make

$$f^{(k-1)}(0) = f^{(k-2)}(0) = \cdots = f'(0) = 0,$$

so that each of these functions remains in absolute value less than or equal to λ throughout the interval, and each satisfies a Lipschitz condition with coefficient λ throughout the interval. The subtraction of this polynomial does not affect the degree of approximation to $f(x)$ which can be obtained by a polynomial of degree n, provided $n \geqq k-1$, and we shall suppose that the subtraction has been performed. With this assumption, it appears from the expression for $g^{(k-1)}(x)$ that a relation holds of the form

$$|g^{(k-1)}(x_2) - g^{(k-1)}(x_1)| \leqq A_k \lambda |x_2 - x_1|,$$

where A_k is a constant dependent only on k. Hence $g(x) = f(\cos x)$ can be represented by a polynomial in $\cos x$ of degree n or lower, and $f(x)$ by such a polynomial in x, with a maximum error not exceeding $A_k K_k \lambda / n^k$.

To make the result applicable in the case of an arbitrary interval, we have merely to note that a linear transformation of the sort already considered, carrying one interval into another, changes the coefficient of the Lipschitz condition for $f^{(k-1)}(x)$ in the inverse ratio of the kth powers of the lengths of the intervals, without otherwise affecting the conditions of the problem.

A summary of the facts in this connection which we are now in a position to recognize may be made as follows:

THEOREM IV.* *For each positive integral value of k there exists a constant L_k having the following property: If $f(x)$ is a function of x possessing a $(k-1)$th derivative which satisfies the Lipschitz condition (5) throughout an interval $a \leqq x \leqq b$, of length l, then there exists for every positive integral value of n,*

* *Thesis*, Satz IV*a*. The restriction $n \geqq k-1$ was overlooked in the thesis; in the process of reasoning, the details of which were dismissed with the words "indem wir wie vorhin die Bedeutung der verschiedenen Grössen überlegen" (p. 39), it was tacitly assumed that $|f(x)|$ itself remains inferior to a constant multiple of λ, which would be true in general only after subtraction of a polynomial of the $(k-1)$th degree. That the theorem is absurd without this restriction is shown by the example $f(x) \equiv x^2$, $k = 3$, $\lambda = 0$, $n = 1$. The restriction does not apply to Theorem III; and the question does not arise with reference to the less definite statement of Theorems II and VII of the thesis. As to the last two propositions, which are contained in IV and III respectively of the present paper, it is seen by comparing the old treatment and the new one that either may be proved independently and used to establish the other.

greater than or equal to $k-1$, a polynomial $\Pi_n(x)$, of the nth degree at most, such that, for all values of x in the interval,

$$|f(x) - \Pi_n(x)| \leqq \frac{L_k l^k \lambda}{n^k}.$$

In particular, in the case $k=2$, the constant $L_2 = \frac{1}{2}K_2$ has this property, if K_2 is a corresponding constant for Theorem III.

It has frequently been noted* that if $f(x)$ is an integrable function of period 2π, and there exists a trigonometric sum $T_n(x)$, of order n or lower, $n \geqq 2$, such that

$$|f(x) - T_n(x)| \leqq \epsilon$$

for all values of x, and if $S_n(x)$ is the partial sum to terms of the nth order of the formal development of $f(x)$ in Fourier's series, then

$$|f(x) - S_n(x)| \leqq K\epsilon \log n,$$

for all values of x, where K is an absolute numerical constant. The argument is, briefly, as follows: Let $f(x)$ be expressed as $T_n(x) + [f(x) - T_n(x)]$. The partial sum of the Fourier's series for $T_n(x)$ is $T_n(x)$ itself, so that an error arises only in the representation of $f(x) - T_n(x)$. As this function never exceeds ϵ in absolute value, its partial sum does not exceed $\epsilon\rho_n$, where

$$(9) \qquad \rho_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \left| \frac{\sin(2n+1)\left(\frac{u-x}{2}\right)}{2\sin\left(\frac{u-x}{2}\right)} \right| du = \frac{2}{\pi}\int_0^{\pi/2} \frac{|\sin(2n+1)t|}{\sin t}\, dt,$$

and the difference between the partial sum and the function can not exceed $\epsilon(1+\rho_n)$. Finally, it is found that, when $n>1$, ρ_n does not exceed a certain multiple† of $\log n$. This fact, in conjunction with Theorems I and III (of which the first may be regarded as a special case of the second), establishes the following result:

THEOREM V.‡ *If $f(x)$ is a function of period 2π possessing a $(k-1)$th*

* See, for example: LEBESGUE, Annales de la Faculté de Toulouse, series 3, vol. 1 (1909), pp. 25–117; pp. 116–117; LEBESGUE, Bulletin de la Soc. Math. de France, vol. 38 (1910), pp. 184–210 (referred to hereafter as LEBESGUE II); pp. 196–197, 201, of the volume, 13–14, 18, of the article. Also, *Thesis*, pp. 49–51.

† For a proof of this well-known fact, see the concluding paragraphs of the present paper.

‡ A somewhat less precise theorem was indicated, but not formally stated, in a recent paper of the author in these Transactions, vol. 13 (1912), pp. 305–318, in a remark following Theorem II of that paper, which is an analogous theorem concerning Legendre's series. The case $k=1$ of the present theorem is treated by a different method in the article LEBESGUE II, pp. 199–201 of the volume, 16–18 of the article. The case $k=2$, less precisely stated, is closely parallel to a statement of WEYL, Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 118–131; p. 128, footnote; but neither includes the other.

derivative which satisfies the Lipschitz condition (5), *and* K *and* K_k *are the numerical constants already so designated, then* $f(x)$ *is everywhere approximately represented by the partial sum to terms of the nth order,* $n > 1$, *of its development in Fourier's series, with an error not exceeding*

$$\frac{KK_k \lambda \log n}{n^k}.$$

II. Numerical Determinations

We turn our attention now to the task of finding numerical values for some of the constants previously left undetermined. Confining ourselves for the present to the case discussed in Theorem I, we suppose a periodic function $f(x)$ given which satisfies the condition (1) for all values of the variables x_1 and x_2, and construct a function $I_m(x)$, which is a trigonometric sum in x of order $2(m-1)$ at most, such that the relation (3) is satisfied for all values of x. We have seen that for positive integral values of m the denominator of the fraction is never less than a certain constant multiple of $1/m$, and the numerator does not exceed a constant multiple of $1/m^2$. We shall undertake to discover something as to the magnitude of these constant multipliers.

Let us set

$$J_m = m \int_0^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^4 du.$$

Making use of the identities

$$\left[\frac{\sin 2u}{\sin u}\right]^4 = 2\cos 4u + 8\cos 2u + 6,$$

$$\left[\frac{\sin 3u}{\sin u}\right]^4 = 2\cos 8u + 8\cos 6u + 20\cos 4u + 32\cos 2u + 19, \tag{10}$$

$$\left[\frac{\sin 4u}{\sin u}\right]^4 = 2\cos 12u + 8\cos 10u + 20\cos 8u + 40\cos 6u + 62\cos 4u + 80\cos 2u + 44,$$

we find the values

$$J_1 = \frac{\pi}{2} = .500\pi, \qquad J_2 = \frac{3\pi}{8} = .375\pi,$$

$$J_3 = \frac{19\pi}{54} = .352\pi -, \qquad J_4 = \frac{11\pi}{32} = .344\pi -.$$

On the other hand, we shall show that

$$\lim_{m=\infty} J_m = \int_0^\infty \frac{\sin^4 u}{u^4} du,$$

and that the value of the expression on the right is $\pi/3$.

To prove the first of these statements, we make a change of variable in the expression defining J_m, and write

$$J_m = \int_0^{m\pi/2} \frac{\sin^4 u}{u^4}\, du + \int_0^{m\pi/2} \sin^4 u \left(\left[m \sin \frac{u}{m} \right]^{-4} - u^{-4} \right) du$$

$$= J_{m1} + J_{m2}.$$

We note for reference the fact, of which we shall make repeated use, that

$$m \sin \frac{u}{m} = u \left(\sin \frac{u}{m} \Big/ \frac{u}{m} \right), \tag{11}$$

and accordingly, if we restrict u to the interval $(0, m\pi/2)$,

$$u \gtreqless m \sin \frac{u}{m} \gtreqless \frac{2}{\pi} u. \tag{12}$$

We proceed to estimate the magnitude of the term J_{m2} above. Assuming that u belongs to the interval of integration, we find

$$m \sin \frac{u}{m} = u \left[1 - \frac{1}{6} \frac{u^2}{m^2} + \cdots \right],$$

$$\left[m \sin \frac{u}{m} \right]^4 = u^4 \left[1 - \frac{4}{6} \frac{u^2}{m^2} + \cdots \right],$$

$$u^4 - \left[m \sin \frac{u}{m} \right]^4 = \frac{4}{6} \frac{u^6}{m^2} + \cdots < a \frac{u^6}{m^2},$$

where a is an absolute constant. Applying this inequality and (12), we have

$$\left[m \sin \frac{u}{m} \right]^{-4} - u^{-4} = \frac{u^4 - m^4 \sin^4 (u/m)}{m^4 u^4 \sin^4 (u/m)} < \left(\frac{\pi}{2} \right)^4 \frac{a}{m^2 u^2} = \frac{a_1}{m^2 u^2},$$

where $a_1 = (\pi/2)^4 a$; and

$$J_{m2} < \frac{a_1}{m^2} \int_0^{m\pi/2} \frac{\sin^4 u}{u^2}\, du.$$

Consequently

$$\lim_{m=\infty} J_{m2} = 0,$$

and

$$\lim_{m=\infty} J_m = \lim_{m=\infty} J_{m1} = \int_0^\infty \frac{\sin^4 u}{u^4}\, du.$$

The value of this expression may be determined by integration by parts:

$$\int_0^\infty \frac{\sin^4 u}{u^4}\, du = \frac{4}{3} \int_0^\infty \frac{\sin^3 u \cos u}{u^3}\, du = \frac{2}{3} \int_0^\infty \frac{3 \sin^2 u \cos^2 u - \sin^4 u}{u^2}\, du$$

$$= \frac{2}{3}\int_0^\infty \frac{3\sin^2 u - 4\sin^4 u}{u^2}\,du;$$

$$\int_0^\infty \frac{\sin^2 u}{u^2}\,du = \frac{\pi}{2},$$

$$(13)\quad \int_0^\infty \frac{\sin^4 u}{u^2}\,du = 4\int_0^\infty \frac{\sin^3 u\cos u}{u}\,du = \int_0^\infty \frac{\sin 2u - \tfrac{1}{2}\sin 4u}{u}\,du$$

$$= \int_0^\infty \frac{\sin u}{u}\,du - \frac{1}{2}\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{4}.$$

Hence

$$\lim_{m=\infty} J_m = \int_0^\infty \frac{\sin^4 u}{u^4}\,du = \frac{2}{3}\left(\frac{3\pi}{2} - \pi\right) = \frac{\pi}{3}.$$

We are led to inquire whether it is universally true that each value of J_m is greater than the following one; it turns out that the answer is in the affirmative. In carrying through the general proof we shall assume that $m \geqq 4$, as we already know that the statement is true for smaller values of m. We may write

$$J_{m+1} - J_m = \int_0^{(m+1)\pi/2}\left[\frac{\sin u}{(m+1)\sin\dfrac{u}{m+1}}\right]^4 du - \int_0^{m\pi/2}\left[\frac{\sin u}{m\sin\dfrac{u}{m}}\right]^4 du$$

$$= \int_{m\pi/2}^{(m+1)\pi/2}\left[\frac{\sin u}{(m+1)\sin\dfrac{u}{m+1}}\right]^4 du$$

$$+ \int_0^{m\pi/2}\sin^4 u\left(\left[(m+1)\sin\frac{u}{m+1}\right]^{-4} - \left[m\sin\frac{u}{m}\right]^{-4}\right)du$$

$$= H_{m1} + H_{m2}.$$

The first of these terms is positive. The second is negative, since the right-hand side of (11) increases as m increases and u/m decreases. We shall show that the negative term is numerically the larger.

Taking first the term H_{m1}, we have within the interval of integration, for $m \geqq 4$,

$$\frac{4}{5}\cdot\frac{\pi}{2} \leqq \frac{m}{m+1}\cdot\frac{\pi}{2} \leqq \frac{u}{m+1} \leqq \frac{\pi}{2},$$

$$\sin\frac{u}{m+1} \geqq \sin\frac{4\pi}{10} > .951,$$

$$1\Big/\sin^4\frac{u}{m+1} < 1.23.$$

By applying this inequality, and then the identity

$$\sin^4 u = \tfrac{1}{8}\cos 4u - \tfrac{1}{2}\cos 2u + \tfrac{3}{8},$$

we obtain the relation

$$H_{m1} < 1.23 \int_{m\pi/2}^{(m+1)\pi/2} \frac{\sin^4 u}{(m+1)^4}\,du = 1.23 \cdot \frac{3\pi}{16} \cdot \frac{1}{(m+1)^4} < \frac{.73}{(m+1)^4}.$$

To form an estimate of the magnitude of the other integral H_{m2}, we need to examine somewhat carefully the behavior of the function $(\sin x)/x$ in the interval $(0, \pi/2)$. If x_1 and x_2 are any two values of the variable.

$$\frac{\sin x_2}{x_2} - \frac{\sin x_1}{x_1} = (x_2 - x_1)\left[\frac{d}{dx}\frac{\sin x}{x}\right]_{x=\xi},$$

where ξ is some value in the interval (x_1, x_2). Now

$$\frac{d}{dx}\frac{\sin x}{x} = \frac{x\cos x - \sin x}{x^2}.$$

In the interval $0 \leqq x \leqq \pi/2$,

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24} < 1 - .397\,x^2,$$

$$x\cos x < x - .397\,x^3,$$

$$\sin x > x - \frac{x^3}{6} > x - .167\,x^3,$$

$$x\cos x - \sin x < -.230\,x^3,$$

$$\frac{d}{dx}\frac{\sin x}{x} < -.230x.$$

If x_1 is less than x_2, then $-\xi$ is less than $-x_1$, and hence

$$\frac{\sin x_2}{x_2} - \frac{\sin x_1}{x_1} < -.230\,x_1(x_2 - x_1).$$

In the interval of integration, u/m and $u/(m+1)$ belong to the interval $(0, \pi/2)$, accordingly

$$m\sin\frac{u}{m} - (m+1)\sin\frac{u}{m+1} = u\left(\frac{\sin(u/m)}{u/m} - \frac{\sin(u/(m+1))}{u/(m+1)}\right)$$

$$< u\left(-.230\frac{u}{m+1}\right)\left(\frac{u}{m} - \frac{u}{m+1}\right) \tag{14}$$

$$= -.230\frac{u^3}{m(m+1)^2}.$$

This may be written

$$1-(m+1)\sin\frac{u}{m+1}\Big/\left(m\sin\frac{u}{m}\right)<-.230\,\frac{u^3}{m(m+1)^2}\Big/\left(m\sin\frac{u}{m}\right).$$

Setting the left-hand side of this relation equal to $1-t$, for the moment, we note that

$$1-t^4=(1-t)(1+t+t^2+t^3)<4(1-t),$$

since $t>1$ and $1-t$ is negative, or, in the old notation,

$$1-\left[(m+1)\sin\frac{u}{m+1}\Big/\left(m\sin\frac{u}{m}\right)\right]^4<-.920\,\frac{u^3}{m(m+1)^2}\Big/\left(m\sin\frac{u}{m}\right),$$

$$\left[m\sin\frac{u}{m}\right]^4-\left[(m+1)\sin\frac{u}{m+1}\right]^4<-.920\,\frac{u^3}{m(m+1)^2}\left[m\sin\frac{u}{m}\right]^3.$$

Hence

$$\left[(m+1)\sin\frac{u}{m+1}\right]^{-4}-\left[m\sin\frac{u}{m}\right]^{-4}<\frac{-.920\,\dfrac{u^3}{m(m+1)^2}}{m\sin\dfrac{u}{m}\left[(m+1)\sin\dfrac{u}{m+1}\right]^4}.$$

Strengthening this inequality still further by means of (12), we recognize finally that

$$H_{m2}<\frac{-.920}{m(m+1)^2}\int_0^{m\pi/2}\frac{\sin^4 u}{u^2}\,du.$$

We have already found (see (13) above) that

$$\int_0^{\infty}\frac{\sin^4 u}{u^2}\,du=\frac{\pi}{4},$$

so that with the assumption that $m\geqq 4$,

$$\int_0^{m\pi/2}\frac{\sin^4 u}{u^2}\,du\geqq\int_0^{2\pi}\frac{\sin^4 u}{u^2}\,du=\frac{\pi}{4}-\int_{2\pi}^{\infty}\frac{\sin^4 u}{u^2}\,du>\frac{\pi}{4}-\int_{2\pi}^{\infty}\frac{du}{u^2}>.625,$$

and

$$H_{m2}<\frac{-.575}{m(m+1)^2}<\frac{-.575}{(m+1)^3}<\frac{-2.8}{(m+1)^4}.$$

Comparing this with the upper limit which was found for the magnitude of H_{m1}, we see that

$$J_{m+1}-J_m=H_{m1}+H_{m2}<0.$$

Since J_m is thus shown to decrease as m increases, and $\lim_{m=\infty} J_m=\pi/3$, we have for all values of m

$$J_m>\frac{\pi}{3}.$$

Now let

$$J'_m = m^2 \int_0^{\pi/2} u \left[\frac{\sin mu}{m \sin u}\right]^4 du.$$

Using once more the identities (10), we find

$$J'_1 = 1.234 - , \qquad J'_2 = .851 - ,$$
$$J'_3 = .778 - , \qquad J'_4 = .745 - . \tag{15}$$

We shall not go so far as to determine whether J'_m always decreases as m increases, but we shall show that, for $m > 4$, $J'_m < .759$.

Following the same procedure as in the case of J_m, we write

$$J'_{m+1} - J'_m = \int_{m\pi/2}^{(m+1)\pi/2} u \left[\frac{\sin u}{(m+1)\sin\frac{u}{m+1}}\right]^4 du$$
$$+ \int_0^{m\pi/2} u \sin^4 u \left(\left[(m+1)\sin\frac{u}{m+1}\right]^{-4} - \left[m \sin\frac{u}{m}\right]^{-4}\right) du$$
$$= H'_{m1} + H'_{m2}.$$

For the first term. we find

$$H'_{m1} < 1.23 \int_{m\pi/2}^{(m+1)\pi/2} \frac{u \sin^4 u}{(m+1)^4} du$$
$$< 1.23 \int_{m\pi/2}^{(m+1)\pi/2} \frac{(m+1)\pi}{2} \cdot \frac{\sin^4 u}{(m+1)^4} du < \frac{1.15}{(m+1)^3}.$$

In the second term, supposing $m \geqq 4$, we apply again most of the work done in connection with H_{m2}, and find

$$H'_{m2} < \frac{-.920}{m(m+1)^2} \int_0^{m\pi/2} \frac{\sin^4 u}{u} du \leqq \frac{-.920}{m(m+1)^2} \int_0^{2\pi} \frac{\sin^4 u}{u} du.$$

For a rough estimate of the value of the last integral, we find

$$\int_0^{2\pi} \frac{\sin^4 u}{u} du = \int_0^{\pi/2} + \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} + \int_{3\pi/2}^{2\pi}$$
$$> \frac{2}{\pi}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}\right) \int_0^{\pi/2} \sin^4 u \, du = \frac{25}{32},$$

and so

$$H'_{m2} < \frac{-.71}{m(m+1)^2} < \frac{-.71}{(m+1)^3}.$$

This inequality, combined with the one previously obtained for H'_{m1}, gives

$$J'_{m+1} - J'_m < \frac{.44}{(m+1)^3}.$$

Repeated application of this relation leads to the result

$$J'_m - J'_4 < .44\left(\frac{1}{5^3}+\frac{1}{6^3}+\cdots+\frac{1}{(m+1)^3}\right) < .44\int_4^\infty \frac{du}{u^3} < .014;$$

and hence, as we have seen that $J'_4 < .745$, it follows that

$$J'_m < .759 \tag{16}$$

for all values of m after the first three.

The results thus obtained give to the inequality (3), for $m \geqq 4$, the form

$$|I_m(x) - f(x)| \leqq 2\lambda \frac{.759}{m^2} \cdot \frac{m}{\pi/3} \leqq \frac{1.45\lambda}{m}.$$

We still have to take account of the circumstance that the index m is not the order of the trigonometric sum $I_m(x)$, which we merely know to be not greater than $2(m-1)$. If n is any positive integer, let m be the integer such that

$$2(m-1) \leqq n < 2m,$$

and let us set

$$I_m(x) = T_n(x).$$

Then $T_n(x)$ is a trigonometric sum of the nth order at most, and as $1/m < 2/n$, we have for $m \geqq 4$, $n \geqq 6$,

$$|f(x) - T_n(x)| \leqq \frac{2.90\lambda}{n}. \tag{17}$$

As for the first five values of n, to which the corresponding values of m are 1, 2, 2, 3, 3 respectively, n is so much less than $2m$ in these cases that the relation (17) may still be obtained as a direct consequence of (15), although (16) no longer holds.

It is obvious that with a little more attention to details it would be possible to replace the constant 2.90 by a somewhat smaller value, even while we keep the same approximation-functions $T_n(x)$: and we have no reason to suppose that these are the best that can be obtained. It is easy, however, to set a limit below which the constant can not be reduced. Suppose that $f(x)$ is the function of period 2π which takes on the values $(-1)^p \pi/(2n+2)$ at the points $2p\pi/(2n+2)$, $p = 0, \pm 1, \pm 2, \cdots$, and is linear in each interval between two successive points of this set. Then $f(x)$ satisfies the Lipschitz condition (1) with $\lambda = 1$. As it is furthermore an even function, we may, in showing that every trigonometric sum of order n or lower must differ from $f(x)$ at some point or other by at least a certain amount, restrict ourselves at the outset to sums of cosines, in consequence of the line of reasoning that was based on the formulæ (4) above. In order to represent $f(x)$ with

an error always less than $\pi/(2n+2)$, a sum $T_n(x)$ must take on positive and negative values alternately at the $n+2$ points $2p\pi/(2n+2)$, $p=0, 1, \cdots, n+1$, and so must vanish at $n+1$ interior points of the interval $(0, \pi)$, — for $n+1$ different values of $\cos x$. As this is impossible for a cosine-sum of the nth order, which is a polynomial of the nth degree in $\cos x$, every trigonometric sum of the nth order or less must differ from $f(x)$ at some point by at least $\pi/(2n+2)$ or *

$$\frac{n}{n+1}\cdot\frac{\pi/2}{n}.$$

The factor $n/(n+1)$ approaches unity when $n=\infty$, and hence the statement of Theorem I surely becomes false if a numerical value smaller than $\pi/2$ is put in place of K_1. Just what is the smallest correct value of K_1 remains undetermined within the limits $(\pi/2, 2.90)$. In view of the fact that the decimal places of the latter number do not even signify a limitation of the method used, but only of the extent to which the calculation was pushed, we may perhaps best sum up our knowledge as follows:

THEOREM VI. *The statement of Theorem I remains correct if the undetermined constant K_1 is replaced by the number 3 (or even by a somewhat smaller number), but not if it is replaced by a numerical value smaller than $\pi/2$.*

We have already seen that the constant L_2 of Theorem II may be taken half as large as K_1. On the other hand, it is easy to show by the construction of particular functions $f(x)$ closely analogous to those introduced just above that L_1 can not be smaller than $\frac{1}{2}$. These facts may be put together as

THEOREM VII. *The constant L_2 of Theorem II may be replaced by the numerical value $\frac{3}{2}$, but not by a value smaller than $\frac{1}{2}$.*

To go one step further, suppose now that the periodic function $f(x)$ has a first derivative $f'(x)$ which everywhere satisfies the condition (7). We have seen how to define a function $I_m(x)$ which is a trigonometric sum of order $2(m-1)$ at most, and which satisfies the relation (6).

Let

$$J_m'' = m^3\int_0^{\pi/2} u^2\left[\frac{\sin mu}{m\sin u}\right]^4 du.$$

The first few values of this quantity are

$$\begin{aligned} J_1'' &= 1.292-, & J_2'' &= .931-, \\ J_3'' &= .914-, & J_4'' &= .868-. \end{aligned} \tag{18}$$

* For a general theorem of which we are here using a very special case, see L. TONELLI, Annali di Matematica, series 3, vol. 15 (1908), pp. 47-119; p. 103; and, for the polynomial case, KIRCHBERGER, *Ueber Tchebychefsche Annäherungsmethoden*, Dissertation, Göttingen, 1902, p. 16.

We shall content ourselves with showing that for larger values of m

$$J''_m < 1.19.$$

We assume $m \geqq 4$, and write

$$J''_{m+1} - J''_m = \int_{m\pi/2}^{(m+1)\pi/2} u^2 \left[\frac{\sin u}{(m+1)\sin\frac{u}{m+1}} \right]^4 du$$

$$+ \int_0^{m\pi/2} u^2 \sin^4 u \left(\left[(m+1)\sin\frac{u}{m+1} \right]^{-4} - \left[m \sin\frac{u}{m} \right]^{-4} \right) du$$

$$= H''_{m1} + H''_{m2}.$$

For the first term,

$$H''_{m1} < 1.23 \int_{m\pi/2}^{(m+1)\pi/2} \frac{u^2 \sin^4 u}{(m+1)^4} du$$

$$< 1.23 \int_{m\pi/2}^{(m+1)\pi/2} \frac{(m+1)^2 \pi^2}{2^2} \cdot \frac{\sin^4 u}{(m+1)^4} du < \frac{1.80}{(m+1)^2}.$$

For the second term,

$$H''_{m2} < \frac{-.920}{m(m+1)^2} \int_0^{m\pi/2} \sin^4 u \, du \leqq \frac{-.920}{m(m+1)^2} \cdot \frac{3m\pi}{16} < \frac{-.54}{(m+1)^2}.$$

Combining the two, we find

$$J''_{m+1} - J''_m < \frac{1.26}{(m+1)^2},$$

when $m \geqq 4$. Repeated application of this inequality gives the result

$$J''_m < J''_4 + 1.26 \int_4^\infty \frac{du}{u^2} < .87 + .32 = 1.19,$$

which is true also when $m = 2$ or 3, by (18).

It follows that

$$| I_m(x) - f(x) | < 4\lambda \frac{1.19}{m^3} \cdot \frac{m}{\pi/3} \leqq \frac{4.55\lambda}{m^2},$$

when $m \geqq 2$. If n is any positive integer, we associate with it a suitable m, by the same convention as in the proof of Theorem VI, and obtain a trigonometric sum $T_n(x)$ of order n or lower, of which it can be affirmed, as $1/m^2 < 4/n^2$ in general and $1/m^2 = 1/n^2$ for $n = 1$, that

$$| f(x) - T_n(x) | \leqq \frac{18.2\lambda}{n^2}.$$

We may state, then, with the understanding that further advance along the same line is surely possible,

THEOREM VIII. *The constant K_2 of Theorem III may be given the value* 20.

Furthermore, since L_2 in Theorem IV may be taken half as large as K_2, we have

THEOREM IX. *The constant L_2 of Theorem IV may be given the value* 10.

In conclusion, we come back to the subject of the constants ρ_n, referred to in connection with Theorem V. Lebesgue* has obtained inequalities that are satisfied by these numbers when n is sufficiently large, and Fejér† has deduced an asymptotic formula which represents them approximately for large values of n. But these relations ‡ do not immediately yield numerical information about the values of ρ_n for specific values of n. It is our next purpose to gain such information, though the results obtained will not be of a high degree of refinement.

If we set

$$j_m = \int_0^{\pi/2} \frac{|\sin mt|}{\sin t}\, dt,$$

then the constants in question, according to (9), are defined by the equation

$$\rho_n = \frac{2}{\pi} j_{2n+1}.$$

Actual integration, with the aid of the identities

$$\frac{\sin 3t}{\sin t} = 1 + 2\cos 2t, \qquad \frac{\sin 5t}{\sin t} = 1 + 2\cos 2t + 2\cos 4t,$$

gives the values

$$j_1 = \frac{\pi}{2} = 1.58-, \qquad j_3 = 2.26-, \qquad j_5 = 2.58-. \tag{19}$$

Transforming the integrals as we have done in similar cases before, let us set

$$j_{m+1} - j_m = \int_{m\pi/2}^{(m+1)\pi/2} \frac{|\sin t|}{(m+1)\sin\dfrac{t}{m+1}}\, dt$$

* LEBESGUE II, pp. 196–198 of the volume, 13–15 of the article.

† FEJÉR, Crelle's Journal, vol. 138 (1910), pp. 22–53; p. 30.

‡ In his recent paper, already referred to in a footnote, GRONWALL has elaborated the theory of these constants to a much greater extent than is attempted here. He obtains the simple relation

$$\rho_n = \frac{4}{\pi^2}\log n + h_n, \qquad \frac{4}{\pi^2} + \frac{9}{4} > h_n > \frac{4C}{\pi^2},$$

C being Euler's constant; this gives a sharper inequality than (20) for large but not for small values of n. He gives also an expansion of ρ_n according to descending powers of $2n+1$ (apart from the logarithmic term), with an estimate of the magnitude of the remainder.

$$+\int_0^{m\pi/2} |\sin t| \left(\left[(m+1) \sin \frac{t}{m+1} \right]^{-1} - \left[m \sin \frac{t}{m} \right]^{-1} \right) dt$$

$$= h_{m1} + h_{m2}.$$

In the first term, if we assume $m \geqq 5$, as we shall do from now on, $\sin \frac{t}{m+1}$ is between $\sin 75° > .965$ and 1, accordingly

$$h_{m1} < \frac{1}{.965} \int_{m\pi/2}^{(m+1)\pi/2} \frac{|\sin t|}{m+1} dt < \frac{1.04}{m+1}.$$

Referring to (14), and using (12) once more, we have the inequality

$$\left[(m+1) \sin \frac{t}{m+1} \right]^{-1} - \left[m \sin \frac{t}{m} \right]^{-1} < -.230 \frac{t}{m(m+1)^2},$$

from which follows

$$h_{m2} < \frac{-.230}{m(m+1)^2} \int_0^{m\pi/2} t |\sin t| dt$$

$$< \frac{-.230}{m(m+1)^2} \left(1 + 2 + \cdots + (m-1)\right) \frac{\pi}{2} \int_0^{\pi/2} |\sin t| dt$$

$$= \frac{-.230}{m(m+1)^2} \cdot \frac{(m-1)m\pi}{4}.$$

As $m \geqq 5$, $(m-1)/(m+1) \geqq \frac{4}{6}$, and

$$h_{m2} < \frac{-.12}{m+1}.$$

Therefore

$$j_{m+1} - j_m = h_{m1} + h_{m2} < \frac{.92}{m+1};$$

$$j_m < j_5 + .92 \left(\frac{1}{6} + \frac{1}{7} + \cdots + \frac{1}{m} \right)$$

$$< j_5 + .92 \int_5^m \frac{du}{u} < 2.58 + .92 \log m - .92 \log 5$$

$$< 1.11 + .92 \log m.$$

We have then for $n \geqq 2$, $m \geqq 5$,

$$\rho_n < \frac{2}{\pi} \left(1.11 + .92 \log (2n+1)\right).$$

Since

$$2n + 1 \leqq \tfrac{5}{2} n,$$

when $n \geqq 2$, it follows that, for such values of n,

$$\rho_n < \frac{2}{\pi}(1.11 + .92 \log n + .92 \log \tfrac{5}{2}) < 1.25 + .59 \log n. \tag{20}$$

It is well known, though of course it does not follow from the work above, that ρ_n becomes infinite with n, and is in fact of the order of magnitude of $\log n$. Fejér showed that the values of ρ_n always increase with n, from a certain point on. He raised the question whether they do so from the very beginning.* It is not difficult to show that this is the case. It is an immediate consequence of (19) that $\rho_0 < \rho_1 < \rho_2$. In order to prove that $\rho_{n+1} > \rho_n$ in general, it will be sufficient to show that the value of the integral j_m increases with m, when $m \geqq 5$.

We write $j_{m+1} - j_m = h_{m1} + h_{m2}$ as before. It is obvious that, since $\sin [t/(m+1)]$ is less than unity,

$$h_{m1} > \frac{1}{m+1}.$$

With reference to h_{m2}, we have to review a considerable part of the work which led up to formula (14). In the interval $(0, \pi/2)$,

$$\cos x > 1 - \frac{x^2}{2},$$

$$x \cos x > x - .500\, x^3.$$

On the other hand,

$$\sin x < x - \frac{x^3}{6} + \frac{x^5}{120} < x - .166\, x^3 + .021\, x^3 = x - .145 x^3,$$

and hence

$$x \cos x - \sin x > - .355\, x^3,$$

$$\frac{d}{dx}\frac{\sin x}{x} > - .355\, x,$$

and, if $0 < x_1 < x_2 < \pi/2$,

$$\frac{\sin x_2}{x_2} - \frac{\sin x_1}{x_1} > - .355\, x_2 (x_2 - x_1).$$

Hence

$$m \sin \frac{t}{m} - (m+1) \sin \frac{t}{m+1} > t\left(- .355 \frac{t}{m}\right)\left(\frac{t}{m} - \frac{t}{m+1}\right)$$

$$= - .355 \frac{t^3}{m^2 (m+1)}.$$

* FEJÉR, loc. cit., p. 31. See references in previous footnotes to the recent paper of GRONWALL.

From this and the second inequality in (12) it follows that

$$\left[(m+1)\sin\frac{t}{m+1}\right]^{-1}-\left[m\sin\frac{t}{m}\right]^{-1}>-.355\frac{\pi^2}{4}\frac{t}{m^2(m+1)}$$

$$>-.877\frac{t}{m^2(m+1)},$$

and so

$$h_{m,2}>\frac{-.877}{m^2(m+1)}\int_0^{m\pi/2}t\,|\sin t|\,dt$$

$$>\frac{-.877}{m^2(m+1)}\frac{\pi}{2}(1+2+\cdots+m)\int_0^{\pi/2}\sin t\,dt$$

$$>\frac{-.69}{m}>\frac{-.83}{m+1},$$

the last inequality holding when $m \geqq 5$. We see thus that j_m does increase with m, and therefore ρ_n with n, from the beginning.

In connection with Theorem V, we were concerned directly with the value of $1+\rho_n$, for which we now have from (20) the inequality

$$1+\rho_n<2.25+.59\log n.$$

The relative importance of the first term of course decreases as n increases. If $n \geqq 5$, then $2.25<1.40\log n$, and $1+\rho_n<2\log n$; if $n \geqq 250$, then $2.25<.41\log n$, so that $1+\rho_n<\log n$. The asymptotic formula of Fejér mentioned above shows that

$$\lim_{n=\infty}\frac{1+\rho_n}{\log n}=\frac{4}{\pi^2}.$$

Of the various results which we are now in a position to formulate, along the line of Theorem V, the following may serve as an example:

THEOREM X.* *If $f(x)$ is a function of period 2π satisfying the Lipschitz condition* (1), *it is represented by the partial sum of its Fourier's series to terms of the nth order, provided $n \geqq 5$, with an error not exceeding $6\lambda(\log n)/n$.*

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* This is an improvement over the result given in the article LEBESGUE II, p. 201 of the volume, 18 of the article; LEBESGUE was not concerned with the reduction of the constant in the numerator to its smallest value, nor with the determination of the point at which the inequality obtained begins to be true.

NOTES AND ERRATA, VOLUMES 7, 13.

VOLUME 7.

Page 337. MAX MASON: *On the boundary value problems of linear ordinary differential equations of second order.*

Equation (22) and the one preceding it are incorrect. The error necessitates the following alterations in order that proper account be taken of the case where $y = const.$ is a solution of the boundary value problem.

Add to the footnote on page 345: In case that $B \equiv 0$, that

$$\int_{x_1}^{x_2} APdx = 0,$$

and that the boundary condition (2_0) may be satisfied by $y = const.$, the condition

$$\int_{x_1}^{x_2} APydx = 0$$

is to be added to the condition expressed by equation (19).

Lines 3, 4, 5 of page 347 should be replaced by:

$$\int_{x_1}^{x_2} APdx = 0,$$

and, as is seen from the expression for J, unless $B \equiv 0$.

The last four lines of page 347 should be replaced by: In this case the condition

$$\int_{x_1}^{x_2} APu_h\,dx = 0$$

is satisfied by the approximating functions, so that from the first equation on this page it follows that

$$\delta_h = \int_{x_1}^{x_2} AP\gamma_h^2\,dx = 1.$$

In the first seven lines of page 348 δ_h should be replaced by unity, and the equation

$$\int_{x_1}^{x_2} AP\bar{u}_h\,dx = 0$$

should be added to line 6 of page 348.

The material from "Now equations (32) ···" in line 18, page 351, to the end of § 2 should be omitted.

On page 352, four lines from the bottom, the words "and it has been shown above that y_0 satisfies (31)" should be omitted.

On page 353 after equation (34) and on page 357 at the end of § 3 the following should be inserted: An exception occurs in the case that $y_0 = const.$ and that

$$\int_{x_1}^{x_2} APdx = 0.$$

VOLUME 13.

Page 352. E. B. VAN VLECK: *On the extension of a theorem of Poincaré* ···.

Line 5. For $(S_1 - R_2)\dfrac{\bar{v}_2(n-1)}{v_1(n-1)}$ read $(S_1 - R_2)\dfrac{\bar{v}_2(n-1)}{\bar{v}_1(n-1)}$.

Page 385. E. B. VAN VLECK: *One-parameter projective groups* ···.

Line 12. For $\dfrac{B}{\rho}\log\dfrac{\rho_4}{\rho_3}$ read $\dfrac{B}{\rho_1}\log\dfrac{\rho_4}{\rho_3}$.